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A Necessary and Sufficient Condition of Output Feedback Stabilizability *

V. Kučera [1] and C. E. de Souza [2]

Abstract

A necessary and sufficient condition for a linear system to be stabilizable via static output feedback is presented. It makes appeal to the linear–quadratic regulator theory.

Introduction

Stabilization of linear systems by constant output feedback gains is a problem that is practically important and theoretically appealing. Conditions for this type of stabilizability essentially involve the Hurwitz or Schur stability of a parametrized family of polynomials (Kučera, 1992), polynomial inequalities (Anderson, Bose and Jury, 1975) or a Lyapunov–like condition (Tsinias and Kalouptsidis, 1990). There is however a different approach which makes appeal to the linear–quadratic regulator theory. It is based on the idea that every stabilizing static state feedback minimizes some quadratic cost functional

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(Kreindler and Jameson, 1972). The output feedback being a special case of the state feedback, stabilizing output gains can be related to special solutions of an algebraic Riccati equation.

There is a body of literature dealing with the search for linear–quadratic optimal output feedback gains (Levine and Athans, 1970; Anderson and Moore, 1971; O'Reilly, 1978; Mäkilä and Toivonen, 1987) or for one with a prescribed degree of sub-optimality (Sezer and Siljak, 1982; Skelton and Xu, 1990). These results are different from ours, however. They seek the output stabilizing gain that minimizes a *given* cost functional whereas we seek any output stabilizing gain making use of the fact that it must minimize *some* cost functional.

Result

We consider a linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

and a static output feedback

$$u = Fy \tag{2}$$

where A, B, C and F are real matrices. We say that system (1) is *output feedback stabilizable* if there exists a feedback control law (2) such that the matrix $A - BFC$ has all its eigenvalues with negative real part. Such matrices will be called stable.

We recall some standard terminology. The pair (A, B) is said to be stabilizable if there exists a real matrix K such that $A - BK$ is stable. The pair (A, C) is said to be detectable if there exists a real matrix L such that $A - LC$ is stable.

Theorem: The system (1) is output feedback stabilizable if and only if

(i) (A, B) is stabilizable and (A, C) is detectable, and

(ii) there exist real matrices F and G such that

$$FC - G = B^T P \tag{3}$$

where P is the real symmetric nonnegative definite solution to

$$A^T P + PA - PBB^T P + C^T C + G^T G = 0. \tag{4}$$

Proof: To prove necessity, suppose that $A - BFC$ is stable for some F. Then (A, B) is stabilizable as $A - BK$ is stable for $K = FC$ and (A, C) is detectable as $A - LC$ is stable for $L = BF$. This proves (i). Since $A - BFC$ is stable there exists (Wonham, 1985, Lemma 12.1) a unique symmetric nonnegative definite matrix P such that

$$(A - BFC)^T P + P(A - BFC) + C^T C + C^T F^T FC = 0. \tag{5}$$

Rearranging (5) we obtain

$$A^T P + PA - PBB^T P + C^T C + (FC - B^T P)^T (FC - B^T P) = 0.$$

Hence, setting $G = FC - B^T P$ implies that (ii) is verified.

To prove sufficiency, suppose that (i) and (ii) hold. It follows from (ii) that equation (5) is satisfied. From (i), the detectability of (A, C) implies that $A - LC$ is stable for some L. Noting that

$$A - LC = (A - BFC) + [-L \quad B] \begin{bmatrix} C \\ FC \end{bmatrix},$$

it follows that $\left(A - BFC, \begin{bmatrix} C \\ FC \end{bmatrix}\right)$ is detectable as well. Since P is symmetric and nonnegative definite, we conclude (Wonham, 1985, Lemma 12.2) from (5) that $A - BFC$ is stable. $\nabla\nabla\nabla$

Interpretation

First of all, our Theorem employs a natural necessary condition, namely (i), for output feedback stabilization to make contact with the linear–quadratic regulator theory

(Kučera, 1972). If $A - BFC$ is stable, then condition (ii) implies that the state feedback gain $K = FC$ minimizes the cost functional

$$J = \int_0^\infty q(x,u)\mathrm{dt} \tag{6}$$

under the constraint (1), where

$$q(x,u) = x^T(C^TC + K^TG + G^TK)x + 2u^TGx + u^Tu.$$

Secondly we note that our Theorem is existentional, i.e. it does not say how to calculate a stabilizing feedback gain F. Nevertheless the relationship with the linear–quadratic regulator is helpful and inspired Geromel and Peres (1985) to propose the following algorithm, which iterates the algebraic Riccati equation (4) up until a weighting matrix is found that satisfies the constraint (3).

Step 1 : Set $i = 0$ and $G_i = 0$.

Step 2 : Solve the equation

$$A^TP_i + P_iA - P_iBB^TP_i + C^TC + G_i^TG_i = 0$$

for P_i symmetric and nonnegative definite.

Step 3 : Set

$$G_{i+1} = B^TP_iC^T(CC^T)^{-1}C - B^TP_i,$$

increase i by 1 and go to Step 2.

If the sequence $P_0, P_1, P_2, ...$ converges, say to P, both (3) and (4) will be satisfied for

$$F = B^TPC^T(CC^T)^{-1}$$

thus yielding one particular output stabilizing gain. The convergence remains to be proved, however.

There are many algorithms that provide an output stabilizing gain F using other principles. Miller, Cochran and Howze (1978) minimize a spectral radius functional. Soh, Berger and Dabke (1985) minimize a cost function based on a simple sufficient condition of stability. Richter and DeCarlo (1984) use homotopy methods for eigenvalue assignment. And finally, Broussard and Halyo (1983) suggest a simple heuristic procedure based on stabilizing perturbations.

Conclusions

We have obtained a new necessary and sufficient condition for output feedback stabilizability of linear systems, which makes appeal to the linear–quadratic regulator theory.

A similar condition was obtained by Trofino–Neto and Kučera (1993). Let E_I and E_K denote the orthogonal projection matrices on $\operatorname{Im} C^T$ and $\operatorname{Ker} C$, respectively. Then the condition reads as follows.

The system (1) is output feedback stabilizable if and only if

(a) (A, B) is stabilizable, and

(b) there exist real matrices $\bar{P}, Q, R$ symmetric positive definite and M such that

$$A^T\bar{P} + \bar{P}A - E_I(B^T\bar{P} + M)^T R^{-1}(B^T\bar{P} + M)E_I + Q = 0$$

and the hamiltonian matrix

$$H = \begin{bmatrix} A - BR^{-1}S & BR^{-1}B^T \\ Q - S^T R^{-1} S & -(A - BR^{-1}S)^T \end{bmatrix}$$

has no pure imaginary eigenvalue for $S = ME_I - B^T\bar{P}E_K$.

This result secures the output feedback structure of the stabilizing gain by replacing the standard algebraic Riccati equation (4) with a modified one. The output stabilizing gain is obtained as

$$F = R^{-1}(B^T\bar{P} + M)C^T(CC^T)^{-1}$$

and the associated state feedback gain $K = FC = R^{-1}(B^T\bar{P} + S)$ minimizes the cost functional (6) under the constraint (1), where this time

$$q(x,u) = x^T Qx + 2u^T Sx + u^T Ru.$$

The conditions (a) and (b) seem to be more difficult to check than the conditions (i) and (ii) of our Theorem.

There is one more advantage of our result. While obtained for continuous–time systems, it can be easily restated for discrete–time systems. However this is not the case for the alternative condition of Trofino–Neto and Kučera (1993) as the discrete–time analogy of the hamiltonian matrix exists under undue restrictions only. A frequency–domain approach (Molinari, 1975) can be used instead.

References

Anderson, B. D. O. and J. B. Moore (1971). *Linear Optimal Control.* Prentice–Hall, Englewood Cliffs, NJ.

Anderson, B. D. O., N. K. Bose and E. I. Jury (1975). Output feedback stabilization and related problems – Solution via decision methods. *IEEE Trans. Automat. Control*, **AC–20**, 53–56.

Broussard, J. R. and N. Halyo (1983). Active flutter control using discrete optimal constrained dynamic compensators. in *Proc. American Control Conf.*, San Francisco, 1026–1034.

Geromel, J. C. and P. L. D. Peres (1985). Decentralised load–frequency control. *IEE Proc.*, **132 D**, 225–230.

Kreindler, E. and A. Jameson (1972). Optimality of linear control systems. *IEEE Trans. Automat. Control*, **AC–17**, 349–351.

Kučera,V. (1972). A contribution to matrix quadratic equations. *IEEE Trans. Automat. Control*, **AC–17**, 344–347.

Kučera, V. (1992). Fixed degree solutions of polynomial equations, in *Prep. 2nd IFAC Workshop on System Structure and Control*, Prague, 24–26.

Levine, W. S. and M. Athans (1970). On the determination of the optimal output feedback gains for linear multivariable systems. *IEEE Trans. Automat. Control*, **AC–15**, 44–48.

Mäkilä, P. M. and H. T. Toivonen (1987). Computational methods for parametric LQ problems – a survey. *IEEE Trans. Automat. Control*, **AC–32**, 658–671.

Miller, L. F., R. G. Cochran and J. W. Howze (1978). Output feedback stabilization by minimization of a spectral radius functional . *Int. J. Control*, **27**, 455–462.

Molinari, B. P. (1975). The stabilizing solution of the discrete algebraic Riccati equation . *IEEE Trans. Automat. Control*, **AC–20**, 396–399.

O'Reilly, J. (1978). Optimal instantaneous output feedback controllers for linear discrete–time systems with inaccessible state. *Int. J. Syst. Sci.*, **9**, 9–16.

Richter, S. and R. DeCarlo (1984). A homotopy method for eigenvalue assignment using decentralized state feedback. *IEEE Trans. Automat. Control*, **AC–29**, 148–158.

Sezer, M. E. and D. D. Siljak (1982). Validation of reduced order models for control design. *J. Guidance Contr.*, bf 5, 5.

Skelton, R. E. and J. H. Xu (1990). Output feedback controllers of suboptimality degree β, *IEEE Trans. Automat. Control*, bf AC–35, 1369–1370 .

Soh, C. B., C. S. Berger and K. P. Dabke (1985). A simple approach to stabilization of discrete–time systems by output feedback. *Int. J. Control*, **42**, 1481–1490.

Trofino–Neto, A. and V. Kučera (1993). Stabilization via static output feedback. *IEEE Trans. Automat. Control*, **AC–38**, 764–765.

Tsinias, J. and N. Kalouptsidis (1990). Output feedback stabilization. *IEEE Trans. Automat. Control*, **AC–35**, 951–954.

Wonham, W. M. (1985). *Linear Multivariable Control: A Geometric Approach.* 3rd ed. Springer, New York.