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EIGENSTRUCTURE ASSIGNMENT IN LINEAR DESCRIPTOR SYSTEMS[1]

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Abstract. Given a linear controllable system $E\dot{x} = Fx + Gu$, we study the limits of linear proportional feedback $u = Kx + v$ in assigning a desired finite and infinite eigenstructure of the closed-loop system $E\dot{x} = (F + GK)x + Gv$. The closed-loop system is not restricted to be regular; its rank makes part of the specifications. Necessary and sufficient conditions are established for the existence of a feedback gain K which assigns the desired eigenstructure, and a procedure is outlined to calculate one such gain.

Keywords. Linear systems; Eigenstructure assignment; State feedback; Controllability; Matrix polynomial equations.

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INTRODUCTION

Let

$$E\dot{x} = Fx + Gu \tag{1}$$

be a linear descriptor system where E, F are $n \times n$ matrices and G is an $n \times m$ matrix of rank m over R, the field of real numbers.

The problem of eigenstructure assignment by linear proportional feedback

$$u = Kx + v \tag{2}$$

where K is an $m \times n$ matrix over R, has had a long history. When E is non-singular, the eigenstructure of (1) is specified by the invariant polynomials of $sE - F$, or equivalently by the position and structure of the system's finite eigenvalues. Then Rosenbrock [12] showed that a *controllable* system (1) with controllability indices $c_1, c_2, ..., c_m$ (decreasingly ordered by magnitude) can be assigned the eigenstructure given by invariant polynomials $\psi_1(s), \psi_2(s), ..., \psi_m(s)$ (decreasingly ordered by degree) if and only if the set of inequalities

$$\sum_{i=j}^{m} \deg \psi_i(s) \;\leq\; \sum_{i=j}^{m} c_i, \quad j = 1, 2, ..., m \tag{3}$$

are satisfied and equality holds when $j = 1$. This famous result, commonly referred to as the fundamental theorem of state feedback, identifies the limits of feedback (2) in altering the eigenstructure of (1). Alternative proofs were given later by Dickinson [1], Flamm [2] and Kučera [5].

A more general situation arises when E is singular. Then the pencil $sE - F$ has finite as well as infinite elementary divisors, or equivalently the system has finite as well as infinite eigenvalues. This means that the system gives rise to exponential as well as impulsive modes. Assuming that $sE - F$ is regular, Kučera and Zagalak [6] generalized Rosenbrock's result in the situation where the desired eigenstructure is specified solely by finite elementary divisors, or by invariant polynomials. The other extreme, where only infinite elementary divisors are being assigned, was considered by Zagalak and Kučera [13].

The problem of simultaneously assigning the finite and the infinite eigenstructure was solved by the same authors in Zagalak and Kučera [14]. The result says that a *regular* controllable system (1) with controllability indices $c_1, c_2, ..., c_m$ (decreasingly ordered by magnitude) can be assigned, by applying feedback (2) that preserves its regularity, the eigenstructure specified by invariant polynomials $\psi_1(s), \psi_2(s), ..., \psi_m(s)$ (decreasingly ordered by degree) and by infinite eigenvalue orders $d_1, d_2, ..., d_m$ (decreasingly ordered by magnitude) if and only if the set of inequalities

$$\sum_{i=j}^{m} \deg \psi_i(s) + d_i \;\leq\; \sum_{i=j}^{m} c_i, \quad j = 1, 2, ..., m \tag{4}$$

are satisfied with equality holding when $j = 1$, and

$$d_i = 0, \quad i > m - q \tag{5}$$

where q is the number of infinite zeros of (1). The necessity of (4) was proved already by Jones, Pugh and Hayton [3] and conditions equivalent to (4) and (5) were given, without proof, in Özçaldiran [10].

The requirement that the resulting feedback system

$$E\dot{x} = (F + GK)x + Gv \tag{6}$$

be regular is too restrictive in many situations. That is why we relax this assumption here and make the rank of $sE - (F + GK)$ a part of the specifications. This will allow for desired eigenstructures specified by a number of invariant polynomials that is less than n.

BASIC CONCEPTS

The system (1) is said to be *controllable* if $\text{rank}[sE - F \quad G] = n$ for all complex s, finite and infinite. For details see Lewis [8] and Verghese, Lvy and Kailath [12] and the references given therein.

Let $N(s), D(s)$ be respectively $n \times m$, $m \times m$ matrices over $R[s]$, the ring of polynomials in s with coefficients in R, such that

$$[sE - F \quad -G] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0.$$

Then $N(s), D(s)$ are said to form a (right) *normal external description* of (1) if

(i) $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is a decreasingly column-degree ordered, minimal polynomial basis of Ker $[sE - F \quad -G]$;

(ii) $N(s)$ is a minimal polynomial basis of $A(sE - F)$, where A is a maximal anihilator of G.

See Malabre, Kučera and Zagalak [9] for details.

Then the *controllability indices* of (1) are the column degrees,

$$c_i = \deg_i \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}, \quad i = 1, 2, ..., m$$

of any normal external description of (1). When $\deg_i D(s) > \deg_i N(s)$ we call c_i a *proper controllability index* of (1).

PROBLEM FORMULATION

Let (1) be a controllable system with controllability indices $c_1 \geq c_2 \geq ... \geq c_m$ and let q be the number of the proper controllability indices. We are given r monic polynomials $\psi_1(s), \psi_2(s), ..., \psi_r(s)$ in $R[s]$ such that $\psi_{i+1}(s)$ divides $\psi_i(s)$, $i = 1, 2, ..., r - 1$, and p positive integers $d_1 \geq d_2 \geq ... \geq d_p$. These polynomials and integers are not arbitrary, however; they are assumed to satisfy

$$\sum_{i=1}^{r} \deg \psi_i(s) \; + \; \sum_{i=1}^{p} d_i \; = \; \sum_{i=1}^{r+m-n} c_i^*. \tag{7}$$

for some selection $c_1^* \geq c_2^* \geq ... \geq c_{r+m-n}^*$ of $r+m-n$ controllability indices of (1) that includes all the proper controllability indices.

The problem of eigenstructure assignment is then as follows. Firstly, establish necesssary and sufficient conditions under which a feedback gain K exists in (2) such that $\psi_i(s)$'s are the invariant polynomials and d_i's are the infinite eigenvalue orders of the closed-loop system (6). Second, if the conditions are verified, give a procedure to calculate one such K.

FUNDAMENTAL THEOREM

The following theorem solves the existence part of the eigenstructure assignment problem.

Theorem. The eigenstructure assignment problem has a solution if and only if

$$n - m + p + q \leq r \leq n \tag{8}$$

and

$$\sum_{i=j}^{r} \deg \psi_i(s) + d_i \ \leq \ \sum_{i=j}^{r} c_i^*, \quad j = 1, 2, ..., r \tag{9}$$

where, by convention, $d_i = 0$ for $i > p$ and $c_i^* = 0$ for $i > r + m - n$.

Proof: (Necessity). We suppose there exists a feedback gain K such that the eigenstructure of (6) is given by the lists $\psi_1(s), \psi_2(s), ..., \psi_r(s)$ and $d_1, d_2, ..., d_p$ that satisfy (7) for some $c_1^*, c_2^*, ..., c_{r+m-n}^*$. We wish to prove (8) and (9).

To handle the finite and the infinite eigenstructure in a uniform way, we shall apply the conformal mapping

$$w = \frac{s}{s-a} \tag{10}$$

where $a \neq 0$ is a complex number that is not an eigenvalue of (6).

Let $N(s), D(s)$ be a normal external description of (1). Then

$$\begin{bmatrix} sE - F - GK & -G \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) - KN(s) \end{bmatrix} = 0 \tag{11}$$

so that $N(s), D(s) - KN(s)$ is a normal external description of the closed-loop system (6). Since controllability is invariant under the action of feedback (2), the systems (1) and (6) are controllable with the same controllability indices.

We define

$$\begin{bmatrix} B(w) \\ C(w) \end{bmatrix} = \begin{bmatrix} N(\frac{aw}{w-1}) \\ D(\frac{aw}{w-1}) - KN(\frac{aw}{w-1}) \end{bmatrix} \text{diag } [(w-1)^{c_1}, ..., (w-1)^{c_m}]. \tag{12}$$

This is a polynomial matrix over $R[w]$ which is irreducible and column proper with column degrees $c_1, c_2, ..., c_m$. Obviously $C(w)$ also has column degrees $c_1, c_2, ..., c_m$. We suppose, without loss of generality, that the matrix E has been brought to the form

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tag{13}$$

with E_1 of full row rank, say f. We define

$$[P(w) \quad -Q(w)] = \mathrm{diag}[\underbrace{w-1,...,w-1}_{f},1,...,1]\left[\tfrac{aw}{w-1}E-(F+GK) \quad -G\right].$$

This is again a polynomial matrix over $R[w]$ which is irreducible and row proper.

It follows from (11) that

$$[P(w) \quad -Q(w)]\begin{bmatrix} B(w) \\ C(w) \end{bmatrix} = 0.$$

Lemma 1 of the Appendix implies that the rank deficiences of $C(w)$ and $P(w)$ are the same, namely

$$\mathrm{rank}\, C(w) = r+m-n,$$

since $\mathrm{rank}P(w) = r$ by assumption. Further the invariant polynomials of $B(w)$ equal those of $Q(w)$, namely

$$\begin{aligned} \varepsilon_1(w) &= 1\,,...,\,\varepsilon_{r+m-n-q}(w)=1, \\ \varepsilon_{r+m-n-q+1}(w) &= w-1\,,...,\,\varepsilon_{r+m-n}(w)=w-1, \\ \varepsilon_{r+m-n+1}(w) &= 1\,,...,\,\varepsilon_m(w)=1 \end{aligned}$$

and the non-unit invariant polynomials of $C(w)$ are equal to those of $P(w)$:

$$\begin{aligned} \phi_1(w) &= \bar{\psi}_1(w)(w-1)^{d_1}\,,...,\,\phi_p(w)=\bar{\psi}_p(w)(w-1)^{d_p}, \\ \phi_{p+1}(w) &= \bar{\psi}_{p+1}(w)\,,...,\,\phi_{r+m-n}(w)=\bar{\psi}_{r+m-n}(w) \end{aligned} \tag{14}$$

where

$$\bar{\psi}_i(w) = (w-1)^{\deg\,\psi_i(s)}\psi_i(\tfrac{aw}{w-1}). \tag{15}$$

Since (12) is irreducible, $\varepsilon_i(w)$ is prime to $\phi_i(w)$, $\;i=1,2,...,r+m-n$. Hence $p \le r+m-n-q$. Since $r \le n$, we have (8).

Now suppose, without loss of generality, that it is the first $r+m-n$ columns of $C(w)$ that are $R(w)$-linearly independent. Let $C'(w)$ be the submatrix of $C(w)$ formed by these columns. Then, by assumption (7), $C'(w)$ is column proper with column degrees $c_1^* \ge c_2^* \ge ... \ge c_{r+m-n}^*$. Then the product $\phi_j(w)...\phi_{r+m-n}(w)$ is the greatest common divisor of all minors of $C'(w)$ of order $r+m-n-j+1$, which implies

$$\sum_{i=j}^{r+m-n} \deg\phi_i(w) \;=\; \sum_{i=j}^{r+m-n} \deg\bar{\psi}_i(w)+d_i \;\le\; \sum_{i=j}^{r+m-n} c_i^*, \quad j=1,...,r+m-n \tag{16}$$

where $d_i = 0$ for $i > p$ has been defined for convenience of summation.

Lemma 1 implies that $\bar{\psi}_i(w) = 1$ for $i = r+m-n+1,...,r$. Adopting the convention that $c_i^* = 0$ for $i > r+m-n$, we can extend the upper summation limit in (16) to r. Finally, since $\deg\bar{\psi}_i(w) = \deg\psi_i(s)$, $\;i=1,2,...,r$ the inequalities (9) are verified.

(Sufficiency). We consider a list of monic polynomials $\psi_1(s), \psi_2(s), ..., \psi_r(s)$ in $R[s]$, a list of positive integers $d_1, d_2, ..., d_p$ and a subset $c_1^*, c_2^*, ..., c_{r+m-n}^*$ of the controllability indices $c_1, c_2, ..., c_m$ that satisfy (7), (8) and (9). We are going to construct a matrix K over R such that the eigenstructure of (6) will be given by these two lists.

To this end, we apply to $\psi_i(s)$ the inverse transformation of (10) to get the polynomials $\bar{\psi}_i(w)$ defined in (15). Clearly $\deg \bar{\psi}_i(w) = \deg \psi_i(s), \quad i = 1, 2, ..., r$. It follows from (9) for $j = r + m - n + 1$ that $\bar{\psi}_i(w) = \psi_i(s) = 1$ for $i > r + m - n$.

Let $N(s), D(s)$ be a normal external description of (1). Applying the same transformation to $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ and $[sE - F \quad -G]$, where E assumes the form (13), we obtain matrices $A(w), B(w)$ that satisfy

$$\operatorname{diag}[\underbrace{w-1, ..., w-1}_{f}, 1, ..., 1] \left[\tfrac{aw}{w-1}E - F \quad -G\right] \begin{bmatrix} B(w) \\ A(w) \end{bmatrix} = 0. \tag{17}$$

Like in (12), $\begin{bmatrix} B(w) \\ A(w) \end{bmatrix}$ is a polynomial matrix over $R[w]$ that is irreducible and column proper with column degrees $c_1, c_2, ..., c_m$.

Now we construct the matrix

$$\bar{C}(w) = \operatorname{diag}[\phi_1(w), ..., \phi_{r+m-n}(w)]$$

where $\phi_i(w)$ are the polynomials defined in (14) and (15). In view of (7) and (9) we can apply to $\bar{C}(w)$ Lemma 2 of the Appendix. There results a matrix $\hat{C}(w)$ having the same invariant polynomials $\psi_1(w), ..., \phi_{r+m-n}(w)$ and that is column proper with column degrees $c_1^* \geq c_2^* \geq ... \geq c_{r+m-n}^*$. We further define the $m \times m$ matrix

$$\tilde{C}(w) = \begin{bmatrix} \hat{C}(w) & 0 \\ 0 & 0 \end{bmatrix}.$$

Let S be a permutation matrix such that the first $r + m - n$ columns of $B(w)S$ have degrees $c_1^*, c_2^*, ..., c_{r+m-n}^*$. Consider the matrix

$$\begin{bmatrix} B(w)S \\ \tilde{C}(w) \end{bmatrix}. \tag{18}$$

If (18) is irreducible, we put $C(w) = \tilde{C}(w)S^{-1}$. If it is not, there must be a zero at $w = 1$ common to $B(w)S$ and $\tilde{C}(w)$. This implies a zero column among the first p columns of (18). As (8) holds and the last $r + m - n - p$ columns of $\tilde{C}(w)$ are linearly independent, there exists a non-singular and constant matrix T such that

$$\begin{bmatrix} B(w)S \\ \tilde{C}(w)T \end{bmatrix}$$

is irreducible. Then we put $C(w) = \tilde{C}(w)TS^{-1}$. In either case, the matrix

$$\begin{bmatrix} B(w) \\ C(w) \end{bmatrix} \tag{19}$$

is irreducible and column proper with column degrees $c_1, c_2, ..., c_m$. Applying the transformation (10) to (19), we obtain the matrix

$$\begin{bmatrix} N(s) \\ H(s) \end{bmatrix} = \begin{bmatrix} B\left(\frac{s}{s-a}\right) \\ C\left(\frac{s}{s-a}\right) \end{bmatrix} \operatorname{diag}\left[(s-a)^{c_1}, ..., (s-a)^{c_m}\right].$$

This matrix is irreducible, column proper and its column degrees are equal to $c_1, c_2, ..., c_m$.

Let us now consider the equation

$$XD(s) + YN(s) = H(s). \tag{20}$$

Since the matrices

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}, \quad \begin{bmatrix} N(s) \\ H(s) \end{bmatrix}$$

are both irreducible and column proper with column degrees $c_1, c_2, ..., c_m$ and $N(s), D(s)$ is a normal external description, the rows of these matrices span the same R-linear spaces. Then, by Lemma 3 of the Appendix, equation (20) has a constant solution pair X, Y such that X is nonsingular. Thus $K = -X^{-1}Y$ qualifies as a feedback gain that assigns the desired eigenstructure to $sE - (F + GK)$. This immediately follows from Lemma 1.

CONSTRUCTION

We summarize the major steps of the sufficiency part of the proof of the Theorem, which provide a construction for K.

(a) Given E, F, G with E having the form (13), calculate polynomial matrices $A(w), B(w)$ such that (17) holds where $B(w)$ and $\begin{bmatrix} B(w) \\ A(w) \end{bmatrix}$ are irreducible and column proper, and $\begin{bmatrix} B(w) \\ A(w) \end{bmatrix}$ is decreasingly column-degree ordered.

(b) Read out $c_1, c_2, ..., c_m$, the column degrees of $\begin{bmatrix} B(w) \\ A(w) \end{bmatrix}$, and identify the subset of q proper controllability indices.

(c) Choose $\psi_1(s), \psi_2(s), ..., \psi_r(s)$ and $d_1, d_2, ..., d_p$ and $c_1^*, c_2^*, ..., c_{r+m-n}^*$ that satisfy (7) and check for the existence of K using (8) and (9).

(d) Construct an $m \times m$ polynomial matrix $C(w)$ having the invariant polynomials $\phi_i(w)$ defined in (14) and (15) and such that the composite matrix $\begin{bmatrix} B(w) \\ C(w) \end{bmatrix}$ is irreducible and column proper with column degrees $c_1, c_2, ..., c_m$.

(e) Find a constant solution pair X, Y of the equation

$$XA(w) + YB(w) = C(w)$$

such that X is non-singular.

(f) Put $K = -X^{-1}Y$.

Of course there may be other feedback gains than K that solve the problem.

CONCLUSIONS

The limits of proportional state feedback (2) in altering the eigenstructure of the linear controllable system (1) have been established in Theorem 1, under the assumption (7) that the total number of the dynamical modes desired is equal to a sum of $r + m - n$ controllability indices. These limits can be summarized as follows.

- **(i)** The eigenvalues can be placed at any position.
- **(ii)** The measure of regularity $r = \text{rank}\,(sE - F - GK)$ is bounded by (8).
- **(iii)** At most $r + m - n$ cyclic chains can be associated with each finite eigenvalue.
- **(iv)** At most $r + m - n - q$ cyclic chains can be associated with the infinite eigenvalue, because q is the number of infinite zero chains.
- **(v)** The sizes of the cyclic chains are limited from below by (9).
- **(vi)** When E is non-singular, all controllability indices are proper, $q = m$, and no infinite eigenvalue can be assigned.

The controllability assumption can be relaxed. A generalization of the Theorem to partially controlable systems is under study. We also conjecture that (7) is a necessary restriction on the eigenstructure desired whenever $r = n$.

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APPENDIX

Lemma 1. [15] Let $P(s), Q(s)$ and $B(s), C(s)$ be polynomial matrices over $R[s]$ of respective size $n \times n,\ n \times m$ and $n \times m,\ m \times m$ such that

$$\begin{bmatrix} P(s) & -Q(s) \end{bmatrix} \begin{bmatrix} B(s) \\ C(s) \end{bmatrix} = 0$$

where $\begin{bmatrix} P(s) & -Q(s) \end{bmatrix}$ and $\begin{bmatrix} B(s) \\ C(s) \end{bmatrix}$ are irreducible, and rank $Q(s) = m$. Then,

the matrices $C(s)$ and $P(s)$ have the same rank deficiency;
the matrices $B(s)$ and $Q(s)$ have the same invariant polynomials;
the matrices $C(s)$ and $P(s)$ have the same non-unit invariant polynomials.

Lemma 2. [15] Let $\bar{C}(s)$ be a column proper, polynomial $k \times k$ matrix over $R[s]$ and let $\phi_1(s), \phi_2(s), ..., \phi_k(s)$ be its invariant polynomials arranged so that $\phi_{i+1}(s)$ divides $\phi_i(s)$, $i = 1, 2, ..., k-1$. Let $c_1 \geq c_2 \geq ... \geq c_k$ be non-negative integers such that

$$\sum_{i=j}^{k} \deg \phi_i(s) \ \leq \ \sum_{i=j}^{k} k_i, \quad j = 1, 2, ..., k$$

with equality holding when $j = 1$.

Then there exist unimodular matrices $U_1(s)$ and $U_2(s)$ such that the matrix

$$\hat{C}(s) = U_1(s)\bar{C}(s)U_2(s)$$

is column proper with column degrees $c_1, c_2, ..., c_k$.

Lemma 3. [7] Let $D(s), N(s)$ and $H(s)$ be respectively $m \times m,\ n \times m$ and $m \times m$ polynomial matrices over $R[s]$. Then the equation

$$XD(s) + YN(s) = H(s)$$

has a solution pair X, Y over R such that X is non-singular if and only if the rows of the matrices

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}, \begin{bmatrix} N(s) \\ H(s) \end{bmatrix}$$

span the same R-linear spaces.