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RESEARCH REPORT

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Dynamics Assignment by PD State Feedback in Linear Reachable Systems

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Dynamics Assignment by PD State Feedback in Linear Reachable Systems [1]

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Abstract. The limits in altering the eigenstructure of linear reachable descriptor systems by proportional-and-derivative (PD) state feedback are studied. Necessary and sufficient conditions are established for a set of invariant polynomials and positive integers to represent the finite and the infinite eigenstructure of a system obtainable from the given descriptor system by PD state feedback. The result implies a constructive procedure to calculate the actual feedback gains.

[1]Based on "Eigenstructure assignment by PD state feedback in linear systems" by P. Zagalak and V. Kučera which appeared in the Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, 11-13 December 1991, pp. 1294-1296. ©IEEE.

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1 INTRODUCTION

Let
$$E\dot{x} = Fx + Gu \tag{1}$$
be a linear descriptor system, where E, F are $n \times n$ matrices, and G is an $n \times m$ matrix of rank m over R, the field of real numbers. We say that (1) is *regular* if $sE - F$ is a non-singular polynomial matrix in s.

We shall study the problem of eigenstructure assignment by proportional-and-derivative (PD) state feedback
$$u = Kx + L\dot{x} + v \tag{2}$$
where K, L are $m \times n$ matrices over R. The resulting closed-loop system
$$(E - GL)\dot{x} = (F + GK)x + Gv \tag{3}$$
is also a descriptor system whose finite and infinite eigenstructure can be described by a list of invariant polynomials and a list of infinite eigenvalue orders, respectively.

Special cases of this problem have been studied in the literature. Rosenbrock [4] obtained the limits of proportional (P) state feedback
$$u = Kx + v \tag{4}$$
in altering the eigenstructure of controllable state-space systems $\dot{x} = Fx + Gu$. Zagalak and Loiseau [6] generalized this result to controllable descriptor systems (1) and showed how the feedback gain K can be calculated for any desired eigenstructure.

The first attempts to establish a similar result for PD state feedback in descriptor systems are due to Shayman [5], Dai [1], and Loiseau [2]. However, they gave only a partial picture of what can be achieved by this type of feedback. Shayman restricted his attention to a constant-ratio PD state feedback. Dai identified the limits of any PD state feedback in assigning only a finite eigenstructure while Loiseau accounted for the infinite eigenstructure as well. In all these cases, however, the maximum number of poles ($= n$) is assigned so that the resulting system is regular.

In this paper we shall generalize the eigenstructure assignment by PD state feedback to the case where less than n poles may be specified and no regularity requirement is imposed. We also offer a new proof which reduces the assignment by PD state feedback to that by P feedback only.

2 BACKGROUND

Let $N(s)$, $D(s)$ be matrices over $R[s]$, the ring of polynomials in the indeterminate s over R, of respective sizes $n \times m$ and $m \times m$ such that
$$\begin{bmatrix} sE - F & -G \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0.$$

Then, $N(s)$ and $D(s)$ are said to form a (right) *normal external description* of (1) if

(i) $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is a decreasingly column-degree ordered, minimal polynomial basis of

Ker $[sE - F \quad -G]$;

(ii) $N(s)$ is a minimal polynomial basis of Ker $P(sE-F)$, where P is a maximal anihilator of G.

Let $N(s), D(s)$ form a normal external description of (1). Denote its column degrees by

$$b_i = \deg_i \, N(s), \quad c_i = \deg_i \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$$

for $i = 1, 2, \ldots, m$. Then it was shown by Malabre, Kučera and Zagalak [3] that c_i, $i = 1, 2, \ldots, m$ are the *controllability indices* of (1) and $r_i = 1 + b_i$, $i = 1, 2, \ldots, m$ are the *reachability indices* of (1). If $c_i = r_i$ then c_i is a *proper* controllability index; otherwise it is called non-proper.

The system (1) is said to be *controllable* if

$$\sum_{i=1}^{m} c_i = \text{rank } E$$

and *reachable* if

$$\sum_{i=1}^{m} r_i = n.$$

3 THE CASE OF P STATE FEEDBACK

We now review the result of Zagalak and Loiseau [6] concerning the eigenstructure assignment for (1) by proportional state feedback (4), resulting in the closed–loop system

$$E\dot{x} = (F + GK)x + Gv. \tag{5}$$

Theorem 1 Let (1) be a controllable system with controllability indices $c_1 \geq c_2 \geq \ldots \geq c_m$ and let q be the number of the proper controllability indices. Let $\psi_1(s), \psi_2(s), ..., \psi_k(s)$ be monic polynomials in $R[s]$ such that $\psi_{i+1}(s)$ divides $\psi_i(s)$, $i = 1, 2, ..., k-1$ and let $d_1 \geq d_2 \geq ... \geq d_p$ be positive integers. Further let $c_1^* \geq c_2^* \geq ... \geq c_{k+m-n}^*$ be the subset of the controllability indices that consists of all proper and the highest-valued non-proper controllability indices.

Then there exists a state feedback (4) that assigns to the system (5) the structure of finite eigenvalues given by $\psi_i(s), i = 1, 2, \ldots, k$ and the infinite eigenvalue structure given by $d_i, i = 1, 2, \ldots, p$ if and only if

$$n - m + p + q \leq k \leq n \tag{6}$$

$$\sum_{i=j}^{k} \deg \psi_i(s) + d_i \leq \sum_{i=j}^{k} c_i^*, \quad j = 1, 2, \ldots, k \tag{7}$$

where, by convention, $d_i = 0$ for $i > p$ and $c_i^* = 0$ for $i > k + m - n$, and equality holds in (7) when $k = n$ and $j = 1$.

4 PROBLEM FORMULATION

Let (1) be a reachable system and let $r_1 \geq r_2 \geq ... \geq r_m$ be its reachability indices. Let $\psi_1(s), \psi_2(s), \ldots, \psi_k(s)$ be monic polynomials such that $\psi_{i+1}(s)$ divides $\psi_i(s)$, $i = 1, 2, \ldots, k-1$ and let $d_1 \geq d_2 \geq \ldots \geq d_p$ be positive integers. Then the problem of eigenstructure assignment by PD state feedback can be stated as follows:

Find necessary and sufficient conditions for a PD state feedback (2) to exist such that the closed-loop system (3) will have the finite eigenvalue structure given by the polynomials $\psi_i(s)$, $i = 1, 2, \ldots, k$ and the infinite eigenvalue structure given by the integers d_i, $i = 1, 2, \ldots, p$.

5 THE CASE OF PD STATE FEEDBACK

The main result of the paper is as follows.

Theorem 2 The problem of eigenstructure assignment by PD state feedback has a solution if and only if

$$n - m + p \leq k \leq n \tag{8}$$

and

$$\sum_{i=j}^{k} \deg \psi_i(s) + d_i \leq \sum_{i=j}^{k} r_i, \quad j = 1, 2, \ldots, k \tag{9}$$

where, by convention, $d_i = 0$ for $i > p$.

Proof: We define first an extended system of (1) by adjoining to the state x its derivative $\dot{x}$. Then,

$$\begin{bmatrix} E & 0 \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} u \ . \tag{10}$$

As the system (1) is reachable, we observe that if $N(s)$, $D(s)$ is a normal external description of (1) then $\begin{bmatrix} N(s) \\ sN(s) \end{bmatrix}$, $D(s)$ is a normal external description of the extended system (10). Moreover, the extended system (10) is controllable with controllability indices r_i, $i = 1, 2, \ldots, m$.

Thus, the action of PD state feedback (2) on the original system (1) can be represented by the action of the pure proportional state feedback

$$u = \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + v \tag{11}$$

upon the extended system (10).

Now the assumptions of Theorem 1 are all satisfied and we can apply Theorem 1 to the extended system (10) and feedback (11). Indeed, the external description reveals that there are no proper controllability indices in (10), and hence the inequality (6) reduces to (8). The inequalities (9) follow immediately from (7).

6 EXAMPLE

Let us consider an ideal integrator

$$\dot{x} = u \tag{12}$$

and analyze the effect of PD state feedback (2),

$$u = Kx + L\dot{x} + v,$$

upon the dynamics of (12). The resulting system is governed by the equation

$$(1 - L)\dot{x} = Kx + v. \tag{13}$$

Both (12) and (13) are reachable systems with reachability index $r_1 = 1$.

We note that $m = n = 1$ so that (8) allows choosing either $k = 1$ (one dynamical mode) or $k = 0$ (no mode at all). The latter case occurs when $K = 0$ and $L = 1$. Then (13) is not regular, the input v is not free and the state x is not uniquely detrmined by the initial condition and the input.

Let us have a closer look at the former case of $k = 1$. There are two further possibilities: either we choose $p = 0$ (one exponential mode) or $p = 1$ (one impulsive mode). One exponential mode is obtained whenever $L \neq 1$; then (13) reads

$$\dot{x} = \frac{K}{1-L}\,x + \frac{1}{1-L}\,v$$

and its invariant polynomial

$$\psi_1(s) = s - \frac{K}{1-L}$$

satisfies (9). We note that any finite eigenvalue can be assigned by choosing K and L appropriately.

One impulsive mode is obtained whenever $L = 1$ and $K \neq 0$; then (13) reduces to the constraint

$$x = -\frac{1}{K}\,v$$

and the integrator input is proportional to the derivative of v. The infinite eigenvalue has order $d_1 = 1$ and (9) is satisfied.

This example illustrates the power of PD state feedback with respect to P state feedback in altering the dynamics of linear reachable systems.

7 CONCLUSIONS

Theorem 2 generalizes the result of Dai [1] and Loiseau [2] on the limits of PD state feedback in altering the eigenstructure of linear reachable descriptor systems. These limits can be summarized as follows.

(i) The measure of regularity k (= the number of desired invariant polynomials) is bounded by (8).

(ii) Eigenvalues can be placed at any position.

(iii) At most $k + m - n$ cyclic chains can be associated with each eigenvalue.

(iv) The sizes of the cyclic chains are limited from below by (9).

We note that reachability is not invariant under PD state feedback; hence the resulting system (3) need not be reachable.

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