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J.J. Hench

The Pseudo-Givens Rotation

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ÚTIA AV ČR P. O. Box 18, 182 08 Prague, Czech Republic
Telex: 122018 atom c, Fax: (+42)(2)6641 4903
e-mail: utia@utia.cas.cz

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The Pseudo-Givens Rotation *

J.J. Hench
Institute of Information and Control Theory
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4
128 08 Praha 8
TEL: (42)(2) 6605-2342
e-mail: hench@utia.cas.cz

Abstract

A pseudo-Givens rotation is a two by two real matrix that has the following properties. First, it zeros an element in an arbitrary column vector from the left, while its inverse zeros an element in an arbitrary row vector from the right. Second, the remaining degrees of freedom are subject to a minimization that is invariant to inversion and proper orthogonal transformation. One of its interesting properties is that the pseudo-Givens rotation is the Givens rotation in the event that the row vector with an element to be zeroed is the transpose of the column vector with an element to be zeroed. Another is that the pseudo-Givens rotation belongs to the set of matrices with determinant equal to one. In this respect, it is similar to the Givens rotation and the Hyperbolic rotation.

Keywords: Orthogonal transformations, Similarity transformations, Givens rotations, Hyperbolic rotations, Non-symmetric matrices.

1 Introduction

The Givens transformation is one of the most basic tools in numerical linear algebra. In fact, many standard algorithms which factor or decompose a

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matrix to canonical form that includes an orthogonal matrix may be based on the Givens rotation [1]. For example, the Givens transformation is a natural tool for the reduction of symmetric matrices to banded form, since there exists a symmetric banded matrix that is orthogonally similar to a full symmetric matrix. In the case where a matrix is nonsymmetric, the orthogonal transformations can zero either its upper or lower bands, but not both simultaneously. Simultaneous reduction of a non-symmetric matrix requires a 2×2 transformation that satisfies constraints in its row and column space. This paper proposes the pseudo-Givens rotation for this purpose. While the pseudo-Givens rotation is not generally orthogonal, some similarities to the Givens transformation remain. Among these is that its determinant is equal to one.

2 Givens Transformation

In this section the Givens rotation is re-examined. The Givens rotation is a 2×2 orthogonal matrix, which may be parameterized by the parameter θ [1]

$$Q = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}. \tag{1}$$

This matrix can be viewed the parameterization of a matrix with a positive determinant and one orthogonality constraint and two normalization constraints; namely, that the two columns (rows) have two-norm equal to one and that the second column (row) is perpendicular to the first. Clearly, then, the only degree of freedom available that remains is the orientation of the first column (row) vector. If we further demand that the following equation is satisfied,

$$Q * u = \alpha e_1$$

where

$$u = \alpha \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix} \quad ; \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2}$$

for some $\alpha \in \mathbb{R}$ with $\theta = 0$ if $\alpha = 0$ and where $\det(Q) = 1$, then these requirements quite naturally lead to the definition of the Givens rotation in (1).

The aim of this section is to show that it is possible to derive the Givens rotation in a different way. Instead of imposing two normalization constraints and an orthogonality constraint, the Givens rotation with be the minimum of a cost function over the domain of real two by two matrices. The criteria for acceptability for the cost functions considered by this paper is that they are invariant to proper orthogonal transformation and inversion. The minimization, therefore, will not favor any particular orientation of the column space or the row space. This is important since the orientation of the Givens rotations in most algorithms is determined by the need to zero an element in an arbitrary vector as in (2).

In the remainder of this section, a cost function is proposed which meets the criteria above. The minimization is designed to minimize the (two–norm) condition number of a matrix. With no external constraints, this minimum will be the Givens rotation. We proceed by proposing these conditions in the following lemma.

Lemma 1 Let $W, \Theta \in \mathbb{R}^{2\times 2}$, such that

$$W = \operatorname{argmin}\left\{\frac{1}{4}\operatorname{trace}\left[\Theta^T\Theta + \Theta^{-T}\Theta^{-1}\right]\right\} \tag{3}$$

subject to the constraint that $Wu = v$ as in (2) and $\det(W) > 0$. There exists a Givens rotation, Q such that $Q = W$.

Proof: Let the singular value decomposition of $\Theta = U\Sigma V^T$. Then,

$$\begin{aligned}\operatorname{trace}\left[\Theta^T\Theta + \Theta^{-T}\Theta^{-1}\right] &= \operatorname{trace}\left[V\Sigma U^T U\Sigma V^T + U\Sigma^{-1}V^T V\Sigma^{-1}U^T\right] \\ &= \sigma_1^2 + \sigma_2^2 + \tfrac{1}{\sigma_1^2} + \tfrac{1}{\sigma_2^2}.\end{aligned} \tag{4}$$

where σ_1 and σ_2 are the singular values of Θ. The global minimum of $\operatorname{trace}\left[\Theta^T\Theta + \Theta^{-T}\Theta^{-1}\right]$ is attained when $\sigma_1 = \sigma_2 = 1$. Since this minimum is attainable with the constraint $Wu = v$, then $W = UV^T$ is an orthogonal matrix. Since W is an orthogonal matrix and since $\det(W) > 0$, W is a Givens rotation. ∎

Remark 1 It is noteworthy that the cost function

$$\frac{1}{4}\operatorname{trace}\left[\Theta^T\Theta + \Theta^{-T}\Theta^{-1}\right]$$

is invariant to proper orthogonal transformations and inversions, since it is equal to the sum of the squares of singular values and their inverses, which is evident in (4).

3 Pseudo-Givens Transformation

The minimization given in the previous section is noteworthy not only in that it is orthogonally invariant, but also that it does not impose an orthogonality condition upon the columns (rows) of the eventual transformation. The motivation for this is as follows. Suppose we are given the constraints on the matrix W such that for an arbitrary vector u

$$\begin{aligned} Wu &= \alpha e_1 \\ u^T W^{-1} &= \alpha e_1^T \end{aligned} \tag{5}$$

with e_1 and α defined as in (2). What seems like two constraints in (5) is actually only one, and the optimal W with respect to the cost function in (4) is a Givens rotation. This type of constraint is found frequently in the reduction of symmetric matrices to similar symmetric banded matrices. Suppose, however, we are given the constraints on the matrix W such that for an arbitrary vectors u and v

$$\begin{aligned} Wu &= \alpha e_1 \\ v^T W^{-1} &= \beta e_1^T. \end{aligned} \tag{6}$$

For arbitrary vectors u and v, it is impossible to insist that the columns (rows) of X are orthogonal. However, it is possible to apply the same minimization procedures from the previous section to compute an W subject to the conditions in (6). First, we elaborate under what conditions such a transformation may be found.

Lemma 2 Let $M, N \in \mathbb{R}^{2\times 2}$ and $\|v\| \neq 0$ and $\|u\| \neq 0$ such that

$$\begin{aligned} Mu &= \alpha e_1 \\ v^T N &= \beta e_1^T \end{aligned} \tag{7}$$

with $\alpha \neq 0$ and $\beta \neq 0$. There exist an M and N such that $M = N^{-1}$ if and only if $v^T u \neq 0$.

Proof: ($\rightarrow$) Suppose $M = N^{-1}$ that satisfies (7), Then

$$\begin{aligned} v^T u &= \alpha\beta e_1^T N^{-1} M^{-1} e_1 \\ &= \alpha\beta e_1^T M M^{-1} e_1 \\ &= \alpha\beta e_1^T e_1 \\ &= \alpha\beta \\ &\neq 0. \end{aligned}$$

($\leftarrow$) Since $\|v\| \neq 0$ and $\|u\| \neq 0$, it is possible to find a Givens rotation Q such that $Qu = \gamma e_1$ with $\gamma \neq 0$. Let $v^T Q^T = w = [\ w_1\ ,\ w_2\ ]$. Since

$$\begin{aligned} v^T u &= v^T Q^T Q u \\ &= \gamma v^T Q^T e_1 \\ &= \gamma w_1, \end{aligned}$$

the condition $v^T u \neq 0$ implies $w_1 \neq 0$. Now, let

$$T = \begin{bmatrix} 1 & \frac{w_2}{w_1} \\ 0 & 1 \end{bmatrix} \tag{8}$$

, $M = TQ$, and $N = T^{-1} Q^T$

$$\begin{aligned} Mu &= TQu \\ &= T\gamma e_1 \\ &= T\gamma e_1 \\ &= \alpha e_1 \end{aligned}$$

where $\alpha = \gamma$. Similarly,

$$\begin{aligned} v^T N &= v^T Q^T T^{-1} \\ &= w^T T^{-1} \\ &= [\ w_1\ ,\ (w_1 w_2 / w_1 - w_2)\ ] \\ &= \beta e_1^T, \end{aligned}$$

where $\beta = w_1$, which completes the proof. ∎

In the previous lemma, the matrix M which satisfies (6) was produced by operating on u first with a Givens rotation to satisfy the first equation and then by finding an upper triangular matrix T to satisfy the second equation. This procedure is used again to find the pseudo-Givens rotation W. Since the minimization procedures are invariant with respect to Givens rotations (*v.* Remark 1), it suffices merely to find the constrained minimum of the cost function whose domain is the set of upper triangular matrices. This minimum is elaborated in the following lemma.

Lemma 3 Let $W, \Theta \in \mathbb{R}^{2\times 2}$, such that

$$W = \operatorname{argmin} \left\{ \frac{1}{4} \operatorname{trace} \left[\Theta^T \Theta + \Theta^{-T} \Theta^{-1} \right] \right\} \tag{9}$$

subject to the constraints

$$\begin{aligned} Wu &= \alpha e_1 \\ v^T W^{-1} &= \beta e_1^T \end{aligned}$$

with $\|v\| \neq 0$, $\|u\| \neq 0$, and $v^T u \neq 0$. Also, let Q be a Givens rotation such that $Qu = \gamma e_1$ with $\gamma \neq 0$ and $v^T Q^T = w$. Then there exists an upper triangular matrix T such that $W = T^{-1} Q$, with

$$T = \begin{bmatrix} \kappa & \frac{-w_2}{\kappa w_1} \\ 0 & \frac{1}{\kappa} \end{bmatrix} \tag{10}$$

where $\kappa = \sqrt[4]{1 + \frac{w_2^2}{w_1^2}}$.

Proof: Since $\|v\| \neq 0$, $\|u\| \neq 0$, and $v^T u \neq 0$, there exists a W that minimizes (9). Also, since the minimization is invariant to Givens rotations, it suffices only to perform the minimization on T, which is given by direct substitution. ∎

Remark 2 It is particularly interesting to note that along with the Givens rotation and the Hyperbolic rotation [1], the resulting pseudo-Givens rotation has determinant $\det(W) = 1$.

4 Conclusion

The pseudo-Givens rotation is elaborated in this paper. It resembles the Givens rotation with the exception that it satisfies constraints placed on both its row and column spaces. The pseudo-Givens rotation is calculated by computing the argument that minimizes a certain cost function which is invariant to inversion and rotation. The resulting pseudo-Givens rotation has a number of interesting properties. First, it is exactly the Givens rotation for symmetric constraints. Second, it belongs to the set of matrices with determinant equal to one. It promises to be of use as an alternative to the Givens rotation in producing similar banded matrices and banded eigenvalue revealing forms for non-symmetric matrices.

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References

[1] Golub, G.H. and C.F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 1983, (second edition: 1989).