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RESEARCH REPORT

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NUMERICAL ANALYSIS OF A NORMALIZED RLS FILTER USING A PROBABILITY DESCRIPTION OF PROPAGATED DATA

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NUMERICAL ANALYSIS OF A NORMALIZED RLS FILTER USING A PROBABILITY DESCRIPTION OF PROPAGATED DATA

J. KADLEC

ABSTRACT. The normalized version of the QR algorithm for recursive least squares estimation and filtering is presented. An understanding of the numerical properties of a normalized RLS algorithm is attempted using a global probability analysis.

KEYWORDS. Systolic array, normalization, fixed point, probability.

1 INTRODUCTION

A normalized version of the QR algorithm [3], [7] for recursive least squares estimation and filtering is presented. All data and parameters of the main triangular section of the array have the guaranteed range of values $[-1, 1]$. The array has the potential for a minimal-latency-implementation, because the normalized section can use DSP, or VLSI, fixed point hardware (look-up-tables) [2].

An understanding of the numerical properties of the normalized RLS algorithm is attempted using a global probability numerical analysis. This approach derives the analytic formulas for the probability density functions (distributions) describing the data (normalized innovations) propagated in the normalized filter.

This approach is used to formulate probability statements about the number of bits used in the fixed point representation of propagated data.

The derived analytic formulas for the probability distributions are verified by the comparison with the data histograms measured on a fixed point normalized array. The array was simulated by C-coded functions under Matlab.

2 NESTED RLS IDENTIFICATION PROBLEMS

We consider the recursive least squares (RLS) identification of a single-output system described by the regression model

$$y(n) = \theta^T(n)\varphi(n) + e(n) \tag{1}$$

where n is discrete time, the p-vector, $\varphi(n)$, is the data regressor, $y(n)$ is the output signal and $e(n)$ represents (in the RLS context) the equation error. The unknown p-vector, θ, of the regression parameters is estimated by the p-vector $\hat{\theta}(n)$. To prepare the ground for the numerical analysis of the algorithms, we will operate with $p(p+1)/2$ different regression models (indexed by $i = 1, 2, ...m-1; m = 2, ...p+1$). Let us denote $\varphi_{1:p+1}(n)$, the vector of data measurements as

$$\varphi^T_{1:p+1}(n) = \left[\underbrace{\varphi^T_{1:i}(n)}_{1\ldots i} \quad \underbrace{\ldots}_{i+1\ldots m-1} \quad \underbrace{\varphi_m(n)}_{m} \quad \underbrace{\ldots}_{m+1\ldots p} \quad \underbrace{y(n)}_{p+1} \right]. \tag{2}$$

The set of RLS models is given by

$$\varphi_m(n) = \theta^T_{1:i|m}(n)\varphi_{1:i}(n) + e_{i|m}(n). \tag{3}$$

The standard, maximal order RLS model (1) is part of the set (3) for $m = p + 1; i = p$.

The estimates $\hat{\theta}_{1:i|m}(n)$ minimize the sum of weighted squares

$$J(\theta_{1:i|m}(n)) = \sum_{j=1}^{j=n} \beta^{2(n-j)} \left(\varphi_m(j) - \theta^T_{1:i|m}(n)\varphi_{1:i}(j)\right)^2 \tag{4}$$

where $0 < \beta^2 \leq 1$ denotes the exponential weighting factor. The sum (4) can be rearranged as

$$J(\theta_{1:i|m}(n)) = \left[\frac{-\theta_{1:i|m}(n)}{1}\right]^T V_{1:i|m}(n) \left[\frac{-\theta_{1:i|m}(n)}{1}\right] \tag{5}$$

where

$$V_{1:i|m}(n) = \sum_{j=1}^{j=n} \beta^{2(n-j)} \left[\frac{\varphi_{1:i}(j)}{\varphi_m(j)}\right] \left[\frac{\varphi_{1:i}(j)}{\varphi_m(j)}\right]^T \tag{6}$$

and this can be computed recursively by

$$V_{1:i|m}(n) = \beta^2 V_{1:i|m}(n-1) + \left[\frac{\varphi_{1:i}(n)}{\varphi_m(n)}\right] \left[\frac{\varphi_{1:i}(n)}{\varphi_m(n)}\right]^T. \tag{7}$$

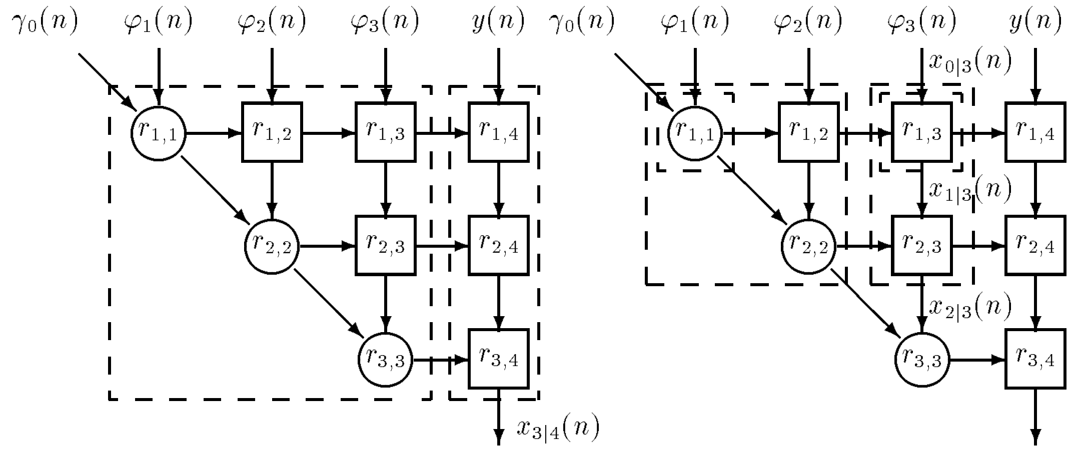
In (7) the two indics i, m indicate the type of data forming the $(i + 1, i + 1)$ matrix $V_{1:i|m}(n)$. The elements of the maximal dimension $(p + 1, p + 1)$ matrix $V(n) = V_{1:p|p+1}(n)$ will be denoted $v_{i,j}(n); i, j = 1, 2, ...p + 1$.

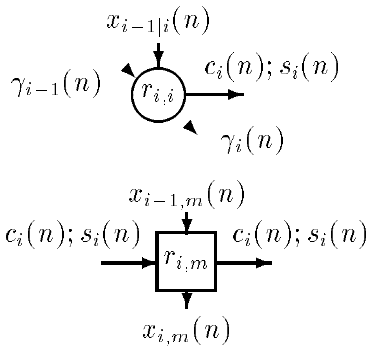
2.1 Nesting in QR Algorithms

We shall deal with algorithms implementing the recursion (7) by a numerically robust factorization of the matrix $V(n)$. The QR algorithm recursively updates the matrix $R(n)$ defined by

$$V(n) \hat{=} R^T(n)R(n). \tag{8}$$

In (8), $R(n)$ is a $(p+1, p+1)$ upper triangular matrix with elements $r_{i,j}(n)$. The a posteriori version of the standard a priori QR update of elements $r_{i,j}(n)$ is well known [3]. The array implementation is summarized in Fig. 1 including the cell descriptions.





$$
\begin{aligned}
r_{i,i}(n) &:= \left(\beta^2 r_{i,i}(n-1) + x^2_{i-1|i}(n)\right)^{1/2}\\
s_i(n) &:= x_{i-1|i}/r_{i,i}(n)\\
c_i(n) &:= \beta r_{i,i}(n-1)/r_{i,i}(n)\\
\gamma_i(n) &:= c_i(n)\gamma_{i-1}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{i,m}(n) &:= \beta\, c_i(n)\, r_{i,m}(n-1) + s_i(n)\, x_{i-1|m}(n)\\
x_{i|m}(n) &:= \left(x_{i-1|m}(n) - s_i(n)\, r_{i,m}(n)\right)/c_i(n)
\end{aligned}
$$

Figure 1. Standard array for RLS identification. Left: Dashed blocks mark the variables related to $\hat{\theta}_{1:3|4}(n)$ filter. Right: Dashed blocks mark 2 RLS subproblems (related to $\hat{\theta}_{1:1|3}(n)$ and $\hat{\theta}_{1:2|3}(n)$) nested in maximal dimension array.

Our numerical analysis will be based on the probability description of the data propagated in the algorithm. To do this we need to understand the relationship of the maximal order QR algorithm (Fig. 1. left) and the nested QR algorithms (Fig. 1. right) related to the estimation of $\theta_{1:i|m}(n)$ by recursively updating the square root factor of $V_{1:i|m}(n)$. This is explained now. Let us consider the QR decompositions of the matrices $V_{1:i|m}(n); m = 2, 3, ...p+1; i = 1, 2, ...m-1$ defined by

$$V_{1:i|m}(n) \hat{=} R^T_{1:i|m}(n) R_{1:i|m}(n). \tag{9}$$

In (9), $R_{1:i|m}(n)$ is a $(i+1, i+1)$ upper triangular matrix.

The following relations (10)-(14) hold as the consequence of the nesting of $V_{1:i|m}(n)$ in $V(n)$ and of the selected $R^T R$ type of factorization:

$$R_{1:i|m}(n) = \left[\begin{array}{ccc|c} r_{1,1}(n) & \dots & r_{i,1}(n) & r_{1,m}(n)\\ & \ddots & \vdots & \vdots\\ & & r_{i,i}(n) & r_{i,m}(n)\\ \hline & & & d_{i|m}(n) \end{array}\right] \tag{10}$$

$$\begin{bmatrix} r_{1,m}(n)\\ \vdots\\ r_{i,m}(n) \end{bmatrix} = \begin{bmatrix} r_{1,1}(n) & \dots & r_{i,1}(n)\\ & \ddots & \vdots\\ & & r_{i,i}(n) \end{bmatrix} \hat{\theta}_{1:i|m}(n) \tag{11}$$

$$\begin{aligned} d^2_{i|m}(n) &= v_{m,m}(n) - [r_{1,m}(n) \dots r_{i,m}(n)][r_{1,m}(n) \dots r_{i,m}(n)]^T \\ &= d^2_{i-1|m}(n) - r^2_{i,m}(n) \qquad (12) \\ d_{m-1|m}(n) &= r_{m,m}(n) \qquad (13) \\ d^2_{0|m}(n) &= v_{m,m}(n). \qquad (14) \end{aligned}$$

Therefore the recursions of all the matrices $R_{1:i|m}(n)$ is already present in the maximal order recursion of elements $r_{i,j}(n)$ forming the matrix $R(n)$. See the example in Fig. 1.

Data $x_{i|m}(n)$ propagated in the array (Fig. 1) are the $\gamma_i(n)$ weighted prediction errors $\hat{e}_{i|m}(n)$

$$x_{i|m}(n) \equiv \hat{e}_{i|m}(n)\gamma_i(n) \qquad (15)$$

where the prediction errors $\hat{e}_{i|m}(n)$ are defined for $m = 1, 2, ...p + 1; i = 1, ..., m - 1$ by

$$\hat{e}_{i|m}(n) \hat{=} \varphi_m(n) - \hat{\theta}^T_{1:i|m}(n-1)\varphi_{1:i}(n)\,;\; \hat{e}_{0|m}(n) \hat{=} \varphi_m(n) \qquad (16)$$

and the angle variables $\gamma_i(n)$ are defined for $i = 1, 2, ..., p$ by

$$\gamma_i(n) \hat{=} \left(1 + \varphi^T_{1:i}(n)\beta^{-2}V^{-1}_{1:i}(n-1)\varphi_{1:i}(n)\right)^{-1/2}\,;\; \gamma_0(n) \hat{=} 1. \qquad (17)$$

The matrix $V_{1:i}(n-1)$ denotes the (i, i) square matrix nested in the uppr left part of the matrix $V(n-1)$.

The scalars $d_{i|m}(n)$ are not directly present in the maximal order QR algorithm. Their time recursion is given by

$$d_{i|m}(n) = \left(\beta^2 d^2_{i|m}(n-1) + x^2_{i|m}(n)\right)^{1/2} \qquad (18)$$

with $x_{0|m}(n) = \varphi_m(n)$. The variables $d_{i|m}(n)$ will be used for the definition of the normalized version of the QR algorithm in the next section.

3 THE NORMALIZED QR ALGORITHM

The coefficients $r_{i,m}(n)$ can be normalized into the range $[-1, 1]$ by suitable scaling factors. This approach was first used by [5] for the normalized RLS Lattice.

Normalized coefficients are defined as

$$\bar{r}_{i,m}(n) = r_{i,m}(n)/d_{i-1|m}(n)\,;\; \bar{r}_{i,i} = 1. \qquad (19)$$

The scaling factors $d_{i|m}(n)$ are used in the definition of the normalized variables:

$$\bar{x}_{i|m} = \hat{e}_{i|m}(n)\gamma_i(n)/d_{i|m}(n) \qquad (20)$$

The transformation of the algorithm from Fig. 1. to the normalized algorithm is allowed using the relation (12). It expresses the fractions resulting from the transformation by the normalized parameters, present in the processors of the normalized array: $d_{i|m}(n)/d_{i-1|m}(n) = \left(1 - \bar{r}^2_{i,m}(n)\right)^{1/2}$. The final normalized systolic array is specified by Fig. 2.

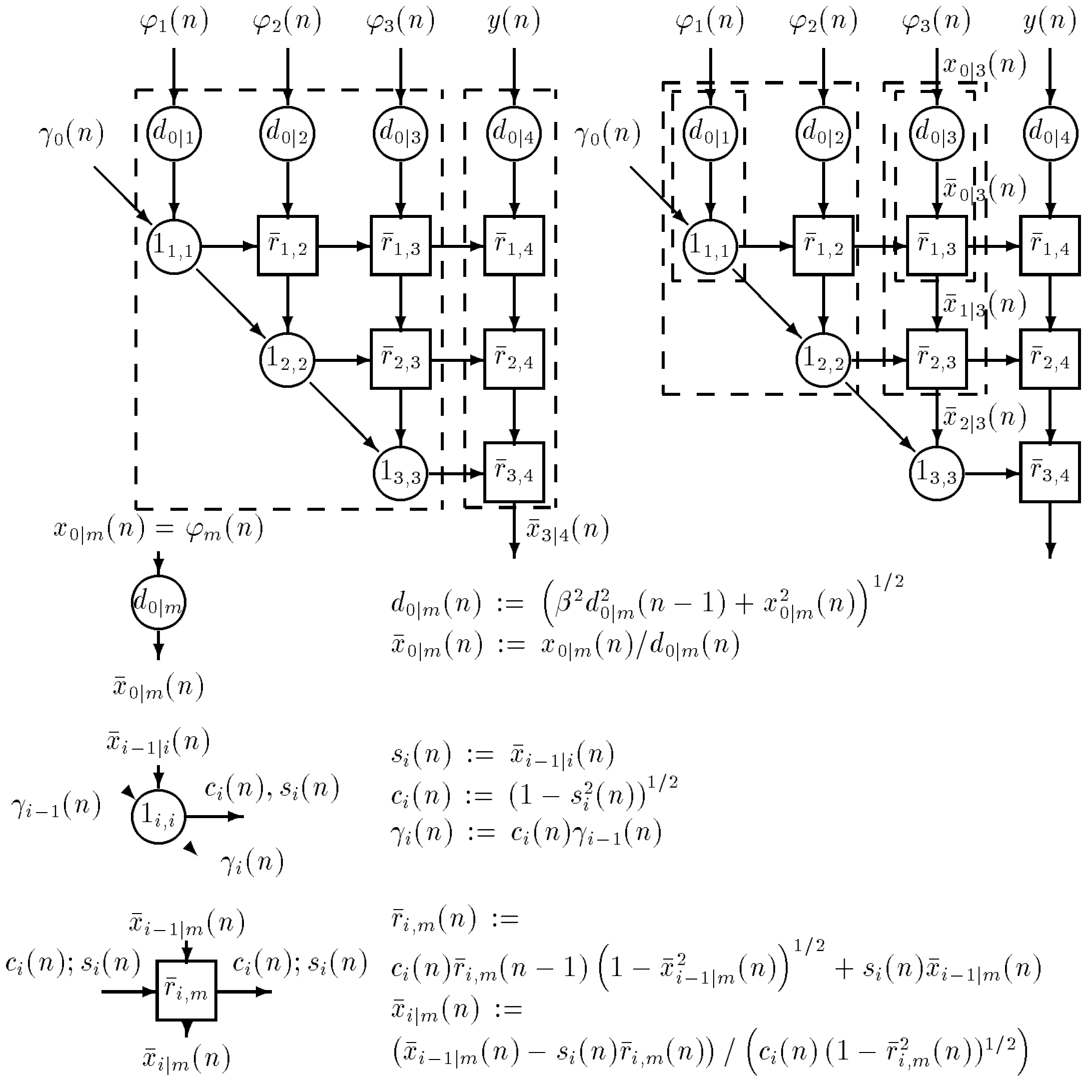


Figure 2. Normalized array for RLS identification. All variables of the triangular section of the normalized array are in the range of values $[-1, 1]$. Dashed boxes present the examples of the RLS subproblems nested in the maximal order array.

The array propagates data $\bar{x}_{i|m}(n) = \hat{e}_{i|m}(n)\gamma_i(n)/d_{i|m}(n)$. With the exception of the input data normalization in the row of processors updating $d_{0|m}(n)$, all the parameters and data propagated in the normalized algorithm have the guaranteed range of values $[-1, 1]$.

4 PROBABILITY IDENTIFICATION

Our objective is to use the concepts of probability identification to analyze the quality of the data representation in the normalized algorithm (Fig. 2.) operating in fixed-point arithmetic with finite precision.

The probability identification [8] works by the recursively updated probability density functions of unknown parameters. To link the general theory with the normalized algorithm and the RLS systolic array implementations we have to make the following assumptions:

4.1 Assumptions

4.1.1 Assumption A Assume that the stochastic part $e_{i|m}(n)$ of models (3) is a zero mean normal white noise, with unknown variance $\sigma^2_{i|m}$

$$\varphi_m(n) = \theta^T_{1:i|m}(n)\varphi_{1:i}(n) + e_{i|m}(n)\,;\; e_{i|m}(n) \sim N(0, \sigma^2_{i|m}). \tag{21}$$

Therefore, the identified systems can be described by

$$p_{i|m}\left(\varphi_m(n)|n-1, \theta_{1:i|m}(n), \sigma^2_{i|m}(n)\right) =$$
$$(2\pi)^{-1/2}\sigma^{-1}_{i|m}(n)\text{exp}\left\{-\frac{1}{2}\sigma^{-2}_{i|m}(n)\left(\varphi_m(n) - \theta^T_{1:i|m}\varphi_{1:i}(n)\right)^2\right\} \tag{22}$$

where $p_{i|m}(|)$ denotes the conditioned probability density function of the random variable φ_m at time (n) conditioned by all the observed data up to the time $n-1$ and by the parameters $\theta_{1:i|m}(n)$, $\sigma^2_{i|m}(n)$.

4.1.2 Assumption B The (prior) probability distribution of unknown parameters at time $n-1$ is assumed to have the Gauss-Wishart distribution

$$p_{i|m}\left(\theta_{1:i|m}(n-1), \sigma^2_{i|m}(n-1)|n-1\right) = k_p(n-1)\sigma^{-(\nu(n-1)+i+2)}_{i|m}(n-1).$$
$$.\text{exp}\left\{-\frac{1}{2}\sigma^{-2}_{i|m}\left[\frac{-\theta_{1:i|m}(n-1)}{1}\right]^T V_{1:i|m}(n-1)\left[\frac{-\theta_{1:i|m}(n-1)}{1}\right]\right\}. \tag{23}$$

4.1.3 Assumption C The parameter tracking is described by the generalized exponential weighting [4]:

$$p_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n-1) =$$
$$k_f(n)\left\{p_{i|m}(\theta_{1:i|m}(n-1), \sigma^2_{i|m}(n-1)|n-1)\right\}^{\beta^2}\left\{q_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n-1)\right\}^{1-\beta^2} \tag{24}$$

where

$$q_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n-1) = k_q\sigma^{-(\nu_q+i+2)}_{i|m}(n).$$
$$.\text{exp}\left\{-\frac{1}{2}\sigma^{-2}_{i|m}(n)\left[\frac{-\theta_{1:i|m}(n)}{1}\right]^T V_q\left[\frac{-\theta_{1:i|m}(n)}{1}\right]\right\} \tag{25}$$

is specified by a scalar $\nu_q > 0$ and $(i+1, i+1)$ matrix $V_q > 0$ selected as $\nu_q \to 0; V_q \to 0$ in order to characterize a flat, noninformative alternative distribution. The weights $k_f(n)$ and k_q are constants weighting the functions to have the integral equal to 1.

4.2 Bayes Formulae

The posterior density of the parameters $p_{i|m}(\theta_{1:i|m}(n), \sigma_{i|m}(n)|n)$ can be computed from the Bayes formulae

$$p_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n) =$$

$$\frac{p_{i|m}(y(n)|n-1, \theta_{1:i|m}(n), \sigma^2_{i|m}(n))}{p_{i|m}(y(n)|n-1)} p_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n-1) \tag{26}$$

where

$$p_{i|m}\left(y(n)|n-1\right) = \int p_{i|m}\left(y(n)|n-1, \theta_{1:i|m}(n), \sigma^2_{i|m}(n)\right).$$
$$.p_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n-1)\mathrm{d}\theta_{1:i|m}(n)\mathrm{d}\sigma^2_{i|m}(n) \tag{27}$$

4.3 Link to RLS

Under the above assumptions the posterior density $p_{i|m}(\theta_{1:i|m}(n), \sigma_{i|m}(n)|n)$ has the reproducing Gauss-Wishart form:

$$p_{i|m}(\theta_{1:i|m}(n), \sigma^2_{i|m}(n)|n) = k_p(n)\sigma_{i|m}^{-(\nu(n)+i+2)}(n).$$
$$.\exp\left\{-\frac{1}{2}\sigma^{-2}_{i|m}(n)\left[\frac{-\theta_{1:i|m}(n)}{1}\right]^T V_{1:i|m}(n)\left[\frac{-\theta_{1:i|m}(n)}{1}\right]\right\} \tag{28}$$

with the recursion of the parameters $V_{1:i|m}(n)$ and $\nu(n)$ given by:

$$V_{1:i|m}(n) = \beta^2 V_{1:i|m}(n-1) + \left[\frac{\varphi_{1:i}(n)}{\varphi_m(n)}\right]\left[\frac{\varphi_{1:i}(n)}{\varphi_m(n)}\right]^T. \tag{29}$$

$$\nu(n) = \beta^2\nu(n-1)+1 \tag{30}$$

This can be derived by substitution of (22) and (23) to (26), (27). See [8] for details.

The time recursion of $V_{1:i|m}(n)$ (29) is identical with the RLS recursion (7). Therefore, both filters from Fig. 1. and Fig. 2. implement the recursion of the factorized matrices, characterizing the probability densities related to the probability identification for all above defined (nested) models.

This is the base for the analysis of the data propagated in the normalized array.

By integration we can derive the mean values of the estimated parameters and dispersions

$$E\left\{\theta_{1:i|m}(n)|n-1\right\} = \hat{\theta}_{1:i|m}(n-1)$$
$$E\left\{\sigma^2_{i|m}(n)|n-1\right\} = \hat{\sigma}^2_{i|m}(n) = \beta^2 d^2_{i|m}(n-1)/(\beta^2\nu(n-1)-2). \tag{31}$$

The distribution of the prediction error $\hat{e}_{i|m}(n)$ can be computed from (22) and (23) by integration (27). It is equal to

$$p(\hat{e}_{i|m}(n)|n-1) = k_e(n)\gamma_i(n)\beta^{-1}d^{-1}_{i|m}(n-1). \tag{32}$$

$$.\left\{1+\hat{e}^2_{1:i|m}(n)\gamma_i^2(n)/(\beta^2 d^2_{i|m}(n-1))\right\}^{-(\beta^2\nu(n-1)+1)/2} \tag{33}$$

where the normalizing constant $k_e(n)$ is given by

$$k_e(n) = \pi^{-1/2}\Gamma\left((\beta^2\nu(n-1)+1)/2\right)/\Gamma\left(\beta^2\nu(n-1)/2\right) \tag{34}$$

and Γ is the Gamma function. The distribution (33) is known as the t-distribution with $E\left\{\hat{e}_{i|m}(n)|n-1\right\} = 0$ and $E\left\{\hat{e}^2_{i|m}(n)|n-1\right\} = \gamma_i^{-2}(n)\hat{\sigma}^2_{i|m}(n)$. Notice, how the uncertainty of the regression coefficients is projected by $\gamma_i^{-2}(n) > 1$ (for $i > 0$) in the increased variance of the prediction error.

5 PROBABILITY ANALYSIS OF THE NORMALIZED ALGORITHM

5.1 Analysis of Normalized Data

The conditioned probability density function $p(\bar{x}_{i|m}(n)|n-1)$ will be derived for the normalized data $\bar{x}_{i|m}(n)$ in this section. The variables $\bar{x}_{i|m}(n)$ are defined as

$$\bar{x}_{i|m}(n) = h\left(\hat{e}_{i|m}(n)\right) = \hat{e}_{i|m}(n)\gamma_i(n)\left\{\beta^2 d^2_{i|m}(n-1) + \hat{e}^2_{i|m}(n)\gamma^2_i(n)\right\}^{-1/2}. \tag{35}$$

The conditional density $p(\hat{e}_{i|m}(n)|n-1)$ specified in (33) is now transformed to the conditional density $p(\bar{x}_{i|m}(n)|n-1)$ using the transformation (35).

In (35), scalars $\beta^2 d^2_{i|m}(n-1)$ and $\gamma^2_i(n)$ are deterministic functions of the data in the condition and can therefore be considered as constants in this transformation.

From the standard lemma about the transformation of random variables it follows that

$$p(\bar{x}_{i|m}(n)|n-1) = p\left(h^{-1}(\bar{x}_{i|m}(n))|n-1\right)\left|\frac{\mathrm{d}h^{-1}\left(\bar{x}_{i|m}(n)\right)}{\mathrm{d}\bar{x}_{i|m}(n)}\right|. \tag{36}$$

The inverse transformation for $-1 < \bar{x}_{i|m}(n) < 1$ is equal to

$$\hat{e}_{i|m}(n) = h^{-1}\left(\bar{x}_{i|m}(n)\right) = \gamma_i^{-1}(n)\beta d_{i|m}(n-1)\bar{x}_{i|m}(n)\left(1-\bar{x}^2_{i|m}(n)\right)^{-1/2} \tag{37}$$

and the Jacobian for $-1 < \bar{x}_{i|m}(n) < 1$ is equal to

$$\left|\frac{\mathrm{d}h^{-1}\left(\bar{x}_{i|m}(n)\right)}{\mathrm{d}\bar{x}_{i|m}(n)}\right| = \gamma_i^{-1}(n)\beta d_{i|m}(n-1)\left(1-\bar{x}^2_{i|m}(n)\right)^{-3/2}. \tag{38}$$

Finally, by substitution into (36), the conditioned density can be found:

$$p(\bar{x}_{i|m}(n)|n-1) = k_e(n)\left(1-\bar{x}^2_{i|m}(n)\right)^{(\beta^2\nu(n-1)-2)/2} \tag{39}$$

where $k_e(n)$ is defined by (34).

To get a clear understanding of the derived result it will be convenient to plot the conditioned densities parametrized only by the exponential weighting factor β^2. This can be done, because the parameter $\nu(n-1)$ converges for a constant β^2 and $n >> 1$ to the values $\nu(n-1) = 1/(1-\beta^2)$ as follows from the time recursion (30). Therefore the final conditioned density parametrized by β^2 is given by

$$p_\beta(\bar{x}_{i|m}(n)|n-1) = \pi^{-1/2}\frac{\Gamma\left((\beta^2/(1-\beta^2)+1)/2\right)}{\Gamma\left(\beta^2/(1-\beta^2))/2\right)}\left(1-\bar{x}^2_{i|m}(n)\right)^{(\beta^2/(1-\beta^2)-2)/2}. \tag{40}$$

The conditioned density $p_\beta(\bar{x}_{i|m}(n)|n-1)$ is plotted in the left part of Fig. 3. Notice, that for $\beta^2 = 0.666$ the density is uniform. Notice also, that for $\beta^2 < 0.666$; the data close to $+1$ and -1 are more and more probable.

5.2 Analysis of Unused Leading Zero Bits

The density functions of the unused bits of the fixed point representation is now derived.

Let us analyze the absolute value of $|\bar{x}_{i|m}(n)|$ only. The density function is given for $0 <$

$|\bar{x}_{i|m}(n)| < 1$ by $p_\beta(|\bar{x}_{i|m}(n)||n-1) = 2p_\beta(\bar{x}_{i|m}(n)|n-1)$.

Let us denote the number of unused leading zero bits in the fixed point representation by a continuous variable b. The continuous variable is used to keep the formulas simple.

The number of unused bits is for $0 < |\bar{x}_{i|m}(n)| < 1$ given by $b = -\log_2\left(|\bar{x}_{i|m}(n)|\right)$.

The inverse transformation for $b > 0$ is given by $|\bar{x}_{i|m}(n)| = 2^{-b}$.

The Jacobian of the transformation is given by $\frac{\mathrm{d}|\bar{x}_{i|m}(n)|}{\mathrm{d}b} = \ln(2)\, 2^{-b}$. Therefore the conditional density of the number of unused leading zero bits is given by

$$p_\beta(b|n-1) = 2\,\pi^{-1/2}\ln(2)\,\frac{\Gamma\left((\frac{\beta^2}{1-\beta^2}+1)/2\right)}{\Gamma\left(\frac{\beta^2}{1-\beta^2}/2\right)}\,2^{-b}\,\left(1-2^{-2b}\right)^{(\frac{\beta^2}{1-\beta^2}-2)/2}. \tag{41}$$

The density parametrized by the β^2 is shown in the right side of Fig. 3.

5.3 Comments

The area below the parametrized density (Fig. 3 right) gives the probability of the numbers of unused leading zero bits in the fixed point data representation.

The derived conditioned density $p_\beta(\bar{x}_{i|m}(n)|n-1)$ is identical for all variables $\bar{x}_{i|m}(n)$; $m = 2, 3, ..., p+1$, $i = 1, 2, ...m-1$ propagated in the normalized identification array.

The result describes for $i = m-1$ the conditional density of the sine variables $s_i(n)$, too (because $s_i(n) = \bar{x}_{i-1|i}(n)$).

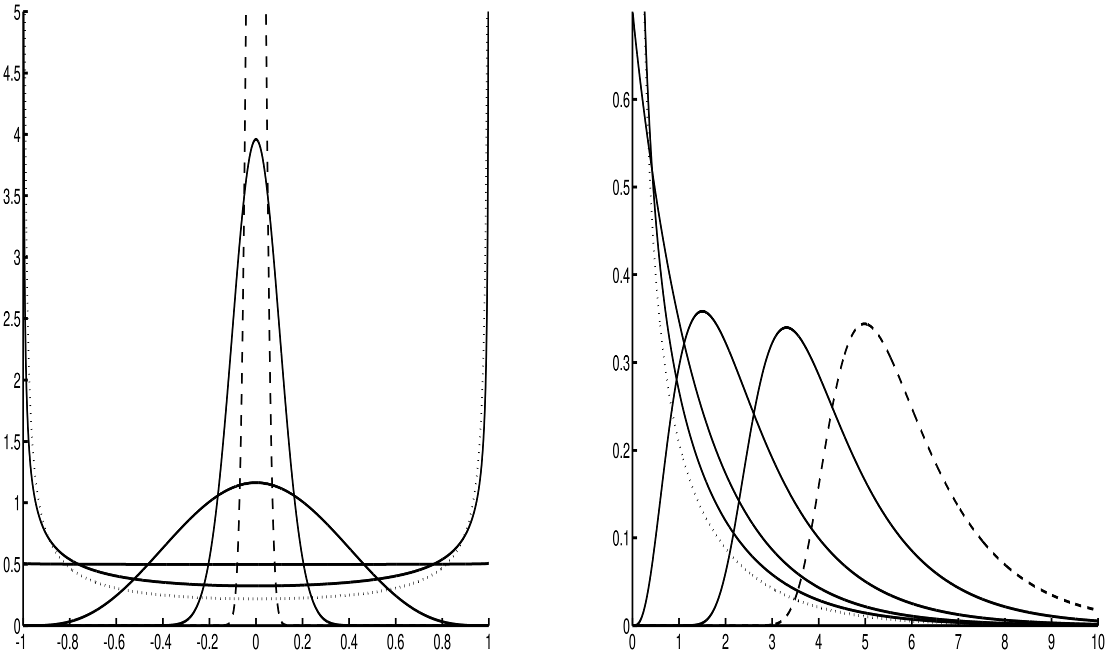
Densities are parametrized by the exponential weighting factor β^2 only. Therefore the value of β^2 is critical for the control of the distribution of data within the range $[-1, 1]$ in the normalized algorithm.

6 COMPUTER SIMULATION

A computer simulation was designed to compare the results of the probability analysis of the normalized algorithms with measured histograms of data, propagated in the fixed point implementation of the normalized algorithm.

A 3-rd order time variable FIR regression model $\theta_{1:3|4}(n)$; $n = 1, \ldots, 1500$ is simulated with the first 2 parameters time variable. The data regressor $\varphi_{1:4}(n)$ is formed by $\varphi^T_{1:4}(n) = [u(n-1), u(n-2), u(n-3), y(n)]$ with white noise input $u(n) \sim N(0, \sigma_u^2)$; $\sigma_u^2 = 0.5$. The normal white noise (added to the simulated system output) is characterized by $N(0, \sigma_u^2)$; $\sigma_u^2 = 0.005$. This data is quantized by 12 bit A/D converters.

The system is identified by the normalized systolic array Fig. 2. for values of exponential weighting equal to $\beta^2 = 0.99$; $\beta^2 = 0.9$ and $\beta^2 = 0.5$.



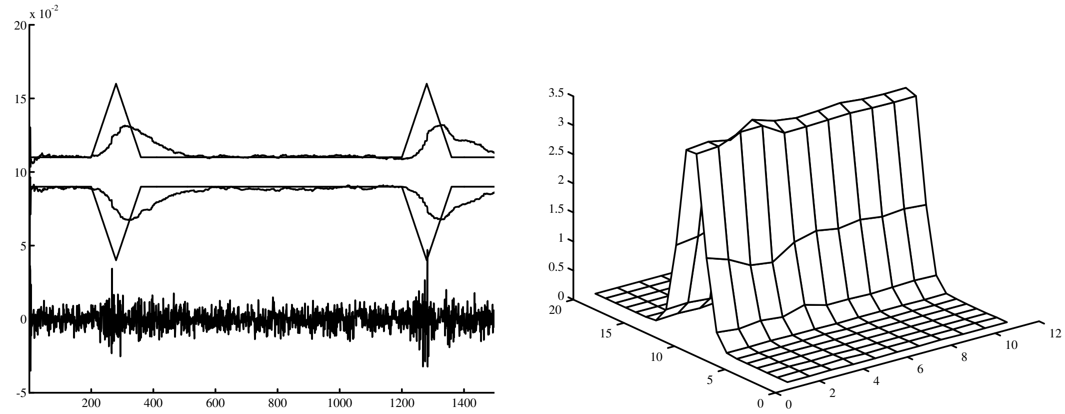
β^2	0.999	0.99	0.9	0.666	0.5	0.333
type of line:	dashed	solid	solid	solid	solid	dotted

Figure 3. Densities of the normalized data $\bar{x}_{i|m}(n)$. Left - probability densities of values. Right - densities of the numbers of lost leading zero bits of the fixed point representation. Lines are parametrized by the exponential weighting factor β^2.

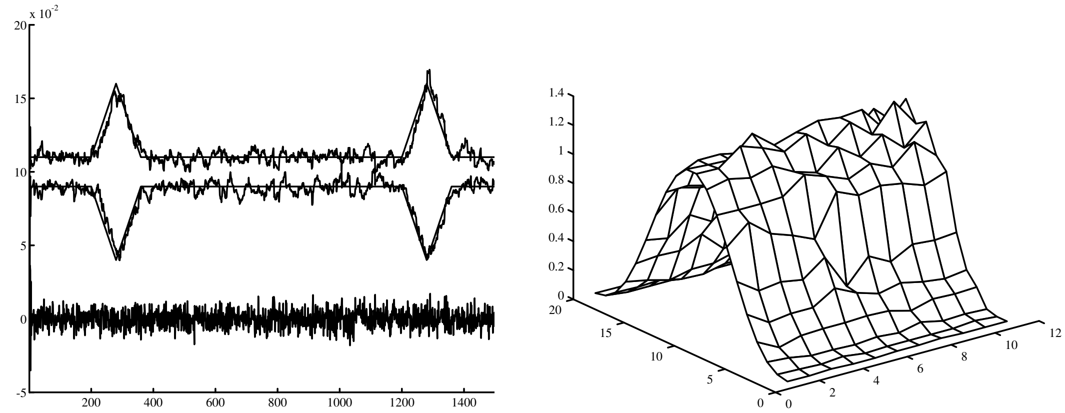
Fig. 4 (left) shows the first 2 elements of the identified vector of regression coefficients $\hat{\theta}_{1:3|4}(n)$ and the prediction errors $\hat{e}_{3|4}(n)$ reconstructed from the normalized data used in the normalized array.

All the normalized variables, normalized parameters and all the intermediate results are computed and represented by 14-bit fixed point arithmetic, in the range $[-1, 1]$. The elementary operations have built in tests to substitute the limit values -1 or 1 if the result is out of the theoretically guaranteed range $[-1, 1]$.

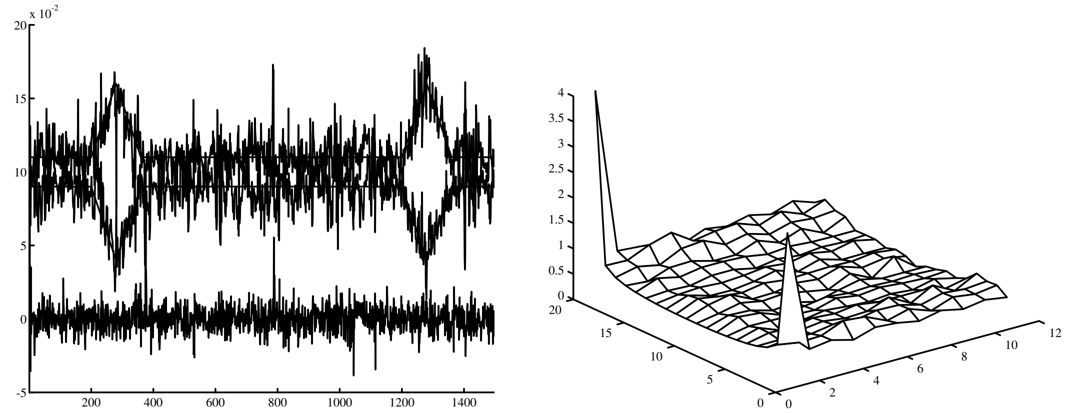
Fig. 4. (right) presents the comparison of the derived analytical distributions (first line) with the histograms (lines 2 - 11) of normalized data $\bar{x}_{i|m}(n)$ measured on a 14-bit fixed-point arithmetic implementation of the normalized array.



$\beta^2 = 0.99$:$\theta_{1:3|4}(n)$; $\hat{\theta}_{1:3|4}(n)$; $\hat{e}_{3|4}(n)$ Analyt. density (1) & real histograms (2-11) of $\bar{x}_{i|m}(n)$.



$\beta^2 = 0.9$:$\theta_{1:3|4}(n)$; $\hat{\theta}_{1:3|4}(n)$; $\hat{e}_{3|4}(n)$ Analyt. density (1) & real histograms (2-11) of $\bar{x}_{i|m}(n)$.



$\beta^2 = 0.5$:$\theta_{1:3|4}(n)$; $\hat{\theta}_{1:3|4}(n)$; $\hat{e}_{3|4}(n)$ Analyt. density (1) & real histograms (2-11) of $\bar{x}_{i|m}(n)$.

Figure 4. Experimental results from the fixed point identification for $\beta^2 = 0.99$; $\beta^2 = 0.9$ and $\beta^2 = 0.5$. The histograms of $\bar{x}_{i|m}(n)$ are compared with the analyticaly derived densities (the first lines in the right side graphs). The lines $2, \ldots, 11$ are the histograms of $\bar{x}_{3|4}$; $\bar{x}_{2|4}$; $\bar{x}_{1|4}$; $\bar{x}_{0|4}$, $\bar{x}_{2|3}$; $\bar{x}_{1|3}$; $\bar{x}_{0|3}$, $\bar{x}_{1|2}$; $\bar{x}_{0|2}$, and $\bar{x}_{0|1}$ measured for $n = 1, \ldots, 1500$ on the 14-bit fixed point implementation of the normalized array.

7 CONCLUSIONS

We have shown how the probability identification can produce information about the quality of the fixed-point data-representation for the normalized systolic identification array.

The analytic formulas for the probability of the number of leading zero bits lost in the fixed point representation of the data have been derived. The computer simulation shows how it compares with the data histograms measured from 14-bit fixed-point implementation of the array.

The Bayesian dimension of the standard RLS identification framework provides important additional information to DSP and VLSI designers, requiring the optimal word-length of fixed-point-arithmetic for normalized RLS identification filters.

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