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RESEARCH REPORT

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Feedback Control of Affine Nonlinear Singular Control Systems

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Feedback Control of Affine Nonlinear Singular Control Systems

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Abstract

This paper discusses feedback control problems like regularization, noninteraction and linearization, for affine nonlinear singular systems. First, based on the constrained dynamic algorithm in affine nonlinear systems, an algorithm is introduced. By using such an algorithm, sufficient and necessary conditions are derived for the solvability of regularization problem. Then, another algorithm is proposed, based on which a sequence of integers can be defined for the system. It is shown that such integers are invariant under the restricted equivalent transformation and the regular feedback. Sufficient and necessary conditions for the solvability of the noninteracting control problem are derived. After that, it is shown that under some conditions, the dynamic part of singular systems can be linearized by using a regular feedback. Finally, an example is provided to illustrate the main results.

Keywords. Singular systems, nonlinear control systems, regularization, noninteracting control, linearization.

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1 Introduction

In this paper, we consider the following affine nonlinear singular systems

$$\begin{aligned} \dot{x} &= f_1(x) + p_1(x)z + g_1(x)u \\ 0 &= f_2(x) + p_2(x)z + g_2(x)u \end{aligned} \tag{1}$$

where $x \in R^n$ is the vector of differential variables, $z \in R^s$ is the vector of algebraic variables, $u \in R^m$ is the vector of inputs, $p_i(x)$ and $g_i(x)$, $i = 1, 2$, are matrix-valued analytic functions having appropriate sizes, $f_1(x)$ and $f_2(x)$ are vector-valued analytic functions with dimensions n and s.

The systems of the form (1) are also known as differential-algebraic systems, see [10]. Though the systems of the form (1) are a special kind of singular nonlinear systems, there exists a large number of physical systems which ought to be modeled according to (1). For instance, contrained robot systems [21], mobile robots [1], constrained mechanical systems [2, 27], chemical processes [10] and so on.

In recent years, there has been substantial progress in dealing with many aspects of singular systems. Some of these successes have been recently listed in books [4] and [7], and in survey papers [3] and [11]. For the case of linear singular systems, many papers and works dealing with the various control problems, which include solvability, controllability, observability, stability, pole-placement, observer design, decoupling, optimal control, etc., have appeared. For a fairly comprehensive introduction to linear singular systems, see books [4] and [7]. For singular nonlinear control systems, a limited attention has been recently devoted to studying the problems like solvability [23, 14, 15], controllability [12], linearization [13, 9], input-output decoupling [16], feedback stabilization [20, 26], Kronecker's canonical forms [24], etc.

In this paper, we mainly discuss the noninteracting control problem for the system (1). The noninteracting control problem is one of the major control design problems, since it aims to reduce a multi-input multi-output system to a set of single-input single-output systems, thus greatly facilitating the control strategy. The problem of noninteracting control has been studied in great detail in regular systems. See Wonham [25], Isidori [8] or Nijmeijer and van der Schaft [22] and references therein for a survey. With regard to the noninteracting control problem of singular systems, the respective published results are limited. The first results on the subject was in the middle of the 1980s [6] for singular linear systems. For the case of singular nonlinear systems, the same problem was investigated in [16]. The results in [16] have the drawback that the conditions under which the noninteracting control problem is solvable depend on an unknown matrix which needs to be chosen first. Related results can be found in [5]

The problem is solved in two steps. First, a generalized constrained dynamic algorithm is used to regularize the system. This means that the system is changed using feedback in order to have a unique solution for any piecewise continuous input and consistent initial state. Regularization is considered as an inevitable part of any feedback design for system (1). Secondly, given a regularizable system, another algorithm is proposed, which produces a sequence of integers. These integers are invariant under the restricted system equivalence and regular proportional state feedback. The second algorithm provides a condition for existence as well as a procedure for construction of a decoupling feedback law. Finally, as an application of the results on the noninteracting control problem, dynamic equation exact linearization problem, i.e. the problem of finding a feedback such that the dynamic part of the corresponding closed-loop system possesses linearity, is addressed.

The paper is organized as follows. In Section 2 we give necessary definitions. Section 3 addresses the regularization problem. The noninteracting control problem is investigated in Section 4. In Section 5, we discuss the problem of dynamic equation exact linearization. Section 6 provides an example to illustrate the main results. Some conclusions and outlooks are contained in Section 5. Long technical proofs are collected in the Appendix.

NOTATIONS. For analytic vector-valued functions $f(x) = [f_1(x), \ldots, f_n(x)]^{\mathrm{T}}$ and $g(x) = [g_1(x), \ldots, g_n(x)]^{\mathrm{T}}$, an analytic function $h(x)$, and an analytic matrix-valued function $p(x) = [p^1(x), \ldots, p^r(x)]$ with $p^i(x) = [p^i_1(x), \ldots, p^i_n(x)]^{\mathrm{T}}$, the following notations will be used in the sequel:

$$\mathrm{d}h(x) = [\tfrac{\partial h(x)}{\partial x_1}, \ldots, \tfrac{\partial h(x)}{\partial x_n}]$$

$$L^0_f h(x) = h(x), L_f h(x) = \mathrm{d}h(x) f(x), L^{k+1}_f h(x) = L_f(L^k_f h(x))$$

$$L_g L_f h(x) = L_g(L_f h(x))$$

$$L_p L^k_f h(x) = [L_{p^1} L^k_f h(x), \ldots, L_{p^r} L^k_f h(x)], \quad k = 0, 1, \ldots$$

where $k = 0, 1 \ldots$, $\mathrm{d}h$ is called as the differential of h, and $L_f h$, the Lie derivative of h along f. See [8, 22] for other differential geometric notations.

2 Definitions and Problem Formulation

Similar to the case of linear systems, the theory of the system (1) depends on what is meant by its solution; throughout this paper we adopt the following definition.

Definition 1 *A pair of functions $x(t)$ and $z(t)$, defined on an interval $[0, T)$ of time, is a solution of the system (1) with initial condition x_0 and piecewise continuous input v if*

1. *$x(t)$ is differentiable and $z(t)$ is piecewise continuous;*
2. *$x(t)$ and $z(t)$ satisfy (1) for all $t \in [0, T)$;*
3. *$x(0) = x_0$.*

Even if the system (1) has solutions for a given initial condition x_0, its solutions need not unique. It is essential in all approaches that a feedback may be chosen so that uniqueness of solutions to the system (1) with x_0 prevails, thus we have the following definition.

Definition 2 *The system (1) is said to be regular at x_0 if it has a unique solution for the initial condition x_0. It is said to be regular on a manifold $M \subset R^n$ if it is regular at any $x_0 \in M$.*

The definitions given above imply some regularity properties of the input u and an appropriate relation between the behavior of $u(0)$ and $z(0)$ and x_0. So in order to lift limitations on the input, we make the following definition.

Definition 3 *The system (1) is said to be strongly regular at x_0 if it has a unique solution for the initial condition x_0 and any piecewise continuous input u. It is said to be strongly regular on a manifold $M \subset R^n$ if it is strongly regular at any $x_0 \in M$.*

In general, the solutions to the system (1) with $x(0) = x_0$ need not unique, even though they exists. Nevertheless in some cases, a feedback may be introduced so that the corresponding closed-loop system has a unique solution for an given initial condition x_0. If that is the case, the system is said to be regularizable. In mathematical terms, regularization problem can be defined as follows.

Definition 4 (**Regularization Problem**) *Find a regular feedback of the form*

$$u = \alpha(x) + \beta(x)v + \gamma(x)z \tag{2}$$

defined in a neighborhood U of x_0 with $\beta(x)$ nonsingular on U, such that the corresponding closed-loop system

$$\begin{array}{rcl} \dot{x} & = & f_1(x) + g_1(x)\alpha(x) + [p_1(x) + g_1(x)\gamma(x)]z + g_1(x)\beta(x)v \\ 0 & = & f_2(x) + g_2(x)\alpha(x) + [p_2(x) + g_2(x)\gamma(x)]z + g_2(x)\beta(x)v \end{array} \tag{3}$$

is strongly regular on U.

Once the question of regularity of a particular system is settled, the closed-loop system may be required to have some additional properties, such as noninteraction, linearity, and so on. The different properties, which the closed-loop system needs to have, correspond to different control problems. In the following, these control problems will be mathematically defined.

Definition 5 (**Noninteracting Control Problem**) *Given an affine nonlinear singular system (1) with output $y = h(x) \in R^n$ and an initial point x_0, find a regular feedback of the form (2), defined in a neighborhood U of x_0 with $\beta(x)$ nonsingular on U, such that the corresponding closed-loop system (3) has the following properties*

(a) it is strongly regular at x_0

(b)

it has noninteracting property. More precisely, the noninteracting property means that for the initial state x_0 and for any pair of input functions $v^a(t)$ and $v^b(t)$ which are defined for $t \geq 0$ and satisfy the condition $(v^a(t))_i = (v^b(t))_i$ for all $t \geq 0$, the corresponding outputs $y^a(t)$ and $y^b(t)$ satisfy $(y^a(t))_i = (y^b(t))_i$ for all $t \geq 0$, while for any i there exist v^a and v^b such that $(y^a)_i \neq (y^b)_i$ if $(v^a)_i \neq (v^b)_i$, where $(\cdot)_i$ denotes the $i-th$ component of $(\cdot)$.

Definition 6 (**Dynamic Equation Exact Linearization Problem**) *Given an affine nonlinear singular system (1) and an initial point x_0, find a regular feedback of the form (2), defined in a neighborhood U of x_0 with $\beta(x)$ nonsingular on U, and a coordinates transformation $\xi = \Phi(x)$ also defined on U, such that the corresponding closed-loop system (3) in the coordinates ξ takes the form*

$$\begin{array}{rcl} \dot{\xi} & = & A\xi + Bv \\ 0 & = & f_2(\xi) + p_2(\xi)z + g_2(\xi)v \end{array} \tag{4}$$

with $p_2(\xi)$ nonsingular and (A, B) controllable.

3 Regularization Problem

This section is devoted to the regularization problem. The following algorithm is based on the constrained dynamic algorithm [2] and the zero dynamic algorithm [3].

Algorithm 1

Step 0. *Assume that $[\,p_2(x) \quad g_2(x)\,]$ has constant rank q_0 in a neighborhood U_0 of x_0 in R^n. If $[\,\bar{p}_2^0(x) \quad \bar{g}_2^0(x)\,]$ is the submatrix formed from the first q_0 independent rows of $[p_2(x) \quad g_2(x)]$, then there exists an $s \times s$ nonsingular matrix $R_0(x)$ such that*

$$R_0(x)\,[\,p_2(x) \quad g_2(x)\,] = \begin{bmatrix} \bar{p}_2^0(x) & \bar{g}_2^0(x) \\ 0 & 0 \end{bmatrix}$$

where $[\,\bar{p}_2^0(x) \quad \bar{g}_2^0(x)\,]$ has q_0 rows. It is convenient to partition $R_0(x)f_2(x)$ as

$$R_0(x)f_2(x) = \begin{bmatrix} \bar{f}_2^0(x) \\ \tilde{f}_2^0(x) \end{bmatrix}$$

where $\bar{f}_2^0(x)$ has q_0 rows.

Premultiplying the algebraic equation of (1) by the matrix $R_0(x)$ gives

$$0 = \begin{bmatrix} \bar{f}_2^0(x) \\ \tilde{f}_2^0(x) \end{bmatrix} + \begin{bmatrix} \bar{p}_2^0(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{g}_2^0(x) \\ 0 \end{bmatrix} u$$

It is obvious that a necessary condition for the existence of solutions to the system (1) is that the initial condition x_0 belongs to the set $M_0 = \{x \in U_0 : \tilde{f}_2^0(x) = 0\}$.

Further assusme the differential $\mathrm{d}\tilde{f}_2^0(x)$ of the mapping $\tilde{f}_2^0(x)$ has constant rank t_0 in a neighborhood U_0 of x_0 . Then locally around x_0, M_0 is an $(n - t_0)$-dimensional submanifold. Without loss of generality, assume that the first t_0 rows of $\mathrm{d}\tilde{f}_2^0(x)$ are independent (otherwise, reorder the rows of $\tilde{f}_2^0(x)$), and denote

$$\tilde{f}_2^0(x) = \begin{bmatrix} \phi_0(x) \\ \tilde{\phi}_0(x) \end{bmatrix}$$

where $\phi_0(x)$ has t_0 rows. Then $M_0 = \{x \in U_0 : \phi_0(x) = 0\}$.

Up to now, the system (1) has been equivalently changed into the following one.

$$\begin{array}{rcl} \dot{x} & = & f_1(x) + p_1(x)z + g_1(x)u \\ 0 & = & \bar{f}_2^0(x) + \bar{p}_2^0(x)z + \bar{g}_2^0(x)u \\ 0 & = & \phi_0(x) \end{array} \tag{5}$$

with $x_0 \in M_0$.

Step 2. *Consider the system (5). Differentiating the last equation of (5) with respect to time and substituting for* $\dot{x}$, *it follows:*

$$0 = L_{f_1}\phi_0(x) + L_{p_1}\phi_0(x)z + L_{g_1}\phi_0(x)u$$

Assume that $\begin{bmatrix} \bar{p}_2^0(x) & \bar{g}_2^0(x) \\ L_{p_1}\phi_0(x) & L_{g_1}\phi_0(x) \end{bmatrix}$ *has constant rank* q_1 *in a neighborhood* U_1 *of* x_0 *in* M_0. *If* $[\bar{p}_2^1(x) \quad \bar{g}_2^1(x)]$ *is the submatrix formed from the first* q_1 *independent rows of* $\begin{bmatrix} \bar{p}_2^0(x) & \bar{g}_2^0(x) \\ L_{p_1}\phi_0(x) & L_{g_1}\phi_0(x) \end{bmatrix}$, *then there exists an* $s \times s$ *nonsingular matrix* $R_1(x)$ *such that*

$$R_1(x)\begin{bmatrix} \bar{p}_2^0(x) & \bar{g}_2^0(x) \\ L_{p_1}\phi_0(x) & L_{g_1}\phi_0(x) \end{bmatrix} = \begin{bmatrix} \bar{p}_2^1(x) & \bar{g}_2^1(x) \\ 0 & 0 \end{bmatrix}$$

Denote

$$R_1(x)\begin{bmatrix} \bar{f}_2^0(x) \\ L_{f_1}\phi_0(x) \end{bmatrix} = \begin{bmatrix} \bar{f}_2^1(x) \\ \tilde{f}_2^1(x) \end{bmatrix}$$

where $\bar{f}_2^1(x)$ *has* q_1 *rows.*

Now it is easily seen that the algebraic equations become as follows:

$$0 = \begin{bmatrix} \bar{f}_2^1(x) \\ \tilde{f}_2^1(x) \end{bmatrix} + \begin{bmatrix} \bar{p}_2^1(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{g}_2^1(x) \\ 0 \end{bmatrix} u$$

It is obvious that a necessary condition for the existence of solutions to the system (1) is that the initial condition x_0 *belongs to the set* $M_1 = \{x \in U_1 : \tilde{f}_2^1(x) = 0\}$.

Further assusme the differential of the mapping $col[\phi_0(x), \tilde{f}_2^1(x)]$ *has constant rank* $t_0 + t_1$ *in a neighborhood* U_1 *of* x_0. *Then locally around* x_0, M_1 *is an* $(n - t_0 - t_1)$*-dimensional submanifold. Without loss of generality, assume that the first* $t_0 + t_1$ *rows of* $col[\mathrm{d}\phi_0(x), \mathrm{d}\tilde{f}_2^1(x)]$ *are independent (otherwise, reorder the rows of* $\tilde{f}_2^1(x)$*), and denote*

$$\tilde{f}_2^1(x) = \begin{bmatrix} \phi_1(x) \\ \tilde{\phi}_1(x) \end{bmatrix}$$

where $\phi_1(x)$ *has* t_1 *rows. Then* $M_1 = \{x \in U_1 : \phi_1(x) = 0\} = \{x \in U_0 : \phi_0(x) = \phi_1(x) = 0\}$.

It is easily seen that if $x_0 \in M_1$, *then the system (1) is equivalent to the following one.*

$$\begin{array}{rcl} \dot{x} & = & f_1(x) + p_1(x)z + g_1(x)u \\ 0 & = & \bar{f}_2^1(x) + \bar{p}_2^1(x)z + \bar{g}_2^1(x)u \\ 0 & = & \phi_1(x) \end{array} \tag{6}$$

with $x_0 \in M_1$.

Step k+1. *Consider the following system*

$$\begin{array}{rcl} \dot{x} & = & f_1(x)+p_1(x)z+g_1(x)u \\ 0 & = & \bar{f}_2^k(x)+\bar{p}_2^k(x)z+\bar{g}_2^k(x)u \\ 0 & = & \phi_k(x) \end{array} \qquad (7)$$

which could be defined by induction, where $x_0 \in M_k$.

Differentiating the last equation of (7) gives

$$0 = L_{f_1}\phi_k(x)+L_{p_1}\phi_k(x)z+L_{g_1}\phi_k(x)u$$

Assume that $\begin{bmatrix} \bar{p}_2^k(x) & \bar{g}_2^k(x) \\ L_{p_1}\phi_k(x) & L_{g_1}\phi_k(x) \end{bmatrix}$ has constant rank q_{k+1} in a neighborhood U_{k+1} of x_0 in M_k. If $[\,\bar{p}_2^{k+1}(x) \quad \bar{g}_2^{k+1}(x)\,]$ is the submatrix formed from the first q_{k+1} independent rows of $\begin{bmatrix} \bar{p}_2^k(x) & \bar{g}_2^k(x) \\ L_{p_1}\phi_k(x) & L_{g_1}\phi_k(x) \end{bmatrix}$, then there exists an $s \times s$ nonsingular matrix $R_{k+1}(x)$ such that

$$R_{k+1}(x)\begin{bmatrix} \bar{p}_2^k(x) & \bar{g}_2^k(x) \\ L_{p_1}\phi_k(x) & L_{g_1}\phi_k(x) \end{bmatrix} = \begin{bmatrix} \bar{p}_2^{k+1}(x) & \bar{g}_2^{k+1}(x) \\ 0 & 0 \end{bmatrix}$$

Denote

$$R_{k+1}(x)\begin{bmatrix} \bar{f}_2^k(x) \\ L_{f_1}\phi_k(x) \end{bmatrix} = \begin{bmatrix} \bar{f}_2^{k+1}(x) \\ \tilde{f}_2^{k+1}(x) \end{bmatrix}$$

where $\bar{f}_2^{k+1}(x)$ has q_{k+1} rows.

Now it is easily seen that the algebraic equations become as follows:

$$0 = \begin{bmatrix} \bar{f}_2^{k+1}(x) \\ \tilde{f}_2^{k+1}(x) \end{bmatrix} + \begin{bmatrix} \bar{p}_2^{k+1}(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{g}_2^{k+1}(x) \\ 0 \end{bmatrix} u$$

It is obvious that a necessary condition for the existence of solutions to the system (1) is that the initial condition x_0 belongs to the set $M_{k+1} = \{x \in U_{k+1} : \tilde{f}_2^{k+1}(x) = 0\}$.

Further assusme the differential of the mapping $col[\phi_0(x), \phi_1(x), \ldots, \phi_k(x), \tilde{f}_2^{k+1}(x)]$ has constant rank $(t_0+t_1+\cdots+t_k+t_{k+1})$ in a neighborhood U_{k+1} of x_0. Then M_{k+1} is an $(n-t_0-t_1-\cdots-t_k-t_{k+1})$-dimensional submanifold. Without loss of generality, assume that the first $t_0+t_1+\cdots+t_k+t_{k+1}$ rows of $col[\mathrm{d}\phi_0(x), \mathrm{d}\phi_1(x), \ldots, \mathrm{d}\phi_k(x), \mathrm{d}\tilde{f}_2^1(x)]$ are independent (otherwise, reorder the rows of $\tilde{f}_2^{k+1}(x)$), and denote

$$\tilde{f}_2^{k+1}(x) = \begin{bmatrix} \phi_{k+1}(x) \\ \tilde{\phi}_{k+1}(x) \end{bmatrix}$$

where $\phi_{k+1}(x)$ has t_{k+1} rows. Then $M_{k+1} = \{x \in U_{k+1} : \phi_{k+1}(x) = 0\} = \{x \in U_0 : \phi_0(x) = \phi_1(x) = \cdots = \phi_k(x) = \phi_{k+1}(x) = 0\}$.

Accordingly, the system (1) has been equivalently changed into the following one.

$$\begin{array}{rcl} \dot{x} & = & f_1(x)+p_1(x)z+g_1(x)u \\ 0 & = & \bar{f}_2^{k+1}(x)+\bar{p}_2^{k+1}(x)z+\bar{g}_2^{k+1}(x)u \\ 0 & = & \phi_{k+1}(x) \end{array} \qquad (8)$$

with $x_0 \in M_{k+1}$.

Remark 1 *The similar algorithm has been used to develop the state space description for differential- algebraic equation systems [10, 26], to solve the disturbance decoupling problem for affine nonlinear singular systems [18], to derive the canonical form for a general class of nonlinear singular systems [24], and to discuss the solvability of general nonlinear singular systems [17].*

It is worth noting that if $\tilde{f}_2^k(x_0) \neq 0$ for some k, then there does not exist any solution $x(t)$ with $x(0) = x_0$ at all. So, the zero-value assumption at every step of Algorithm 1 is necessary for the system (1) to have a solution. If at every step of Algorithm 1 the two constant rank assumptions and the zero-value assumption are satisfied, then we call x_0 a regular point of Algorithm 1.

Remark 2 *Observe that the sequence t_k is decreasing and bounded by 0, i.e.*

$$t_1 \geq t_2 \geq \cdots \geq t_{k+1} \geq \cdots \geq 0$$

and the sequence q_k is increasing and bounded by s, i.e.

$$q_1 \leq q_2 \leq \cdots \leq q_{k+1} \leq \cdots \leq s$$

Therefore, Algorithm 1 will terminate after a finite number of steps, say k^*+1 (bounded by n), with $t_{k^*} > 0$ and $t_{k^*+1} = 0$. In addition, by construction, it follows that

$$t_0 \leq s - q_0, \quad t_{k+1} \leq t_k - q_{k+1} + q_k, \quad k > 0 \tag{9}$$

The following properties of the algorithm are very useful in discussing the regularization problem.

Lemma 1 *The integers q_i and t_i, $i = 0, ...,$ and the vectors $\tilde{f}_2^i(x)$ and $\phi_i(x)$, $i = 0, ...,$ are invariant under the feedback of the type (2).*

Proof: The proof is by induction on the steps of the algorithm. For $i = 0$, the result is obvious. Assuming that the result holds for $i = k$, then it is easily seen that the following relation is satisfied.

$$\begin{aligned}\begin{bmatrix} \bar{p}_2^k(x) & \bar{g}_2^k(x) \\ L_{p_1}\phi_k(x) & L_{g_1}\phi_k(x) \end{bmatrix} &\rightarrow \begin{bmatrix} \bar{p}_2^k(x) + \bar{g}_2^k(x)\gamma(x) & \bar{g}_2^k(x)\beta(x) \\ L_{p_1}\phi_k(x) + L_{g_1}\phi_k(x)\gamma(x) & L_{g_1}\phi_k(x)\beta(x) \end{bmatrix} \\ &= \begin{bmatrix} \bar{p}_2^k(x) & \bar{g}_2^k(x) \\ L_{p_1}\phi_k(x) & L_{g_1}\phi_k(x) \end{bmatrix} \begin{bmatrix} I & 0 \\ \gamma(x) & \beta(x) \end{bmatrix}\end{aligned}$$

which implies that $R_{k+1}(x) \rightarrow R_{k+1}(x)$ and $q_{k+1} \rightarrow q_{k+1}$ because of the nonsingularity of $\beta(x)$ where $\rightarrow$ denotes the action of feedback. In addition,

$$\begin{aligned}\begin{bmatrix} \bar{f}_2^{k+1}(x) \\ \tilde{f}_2^{k+1}(x) \end{bmatrix} = R_{k+1}(x) \begin{bmatrix} \bar{f}_2^k(x) \\ L_{f_1}\phi_k(x) \end{bmatrix} &\rightarrow R_{k+1}(x) \begin{bmatrix} \bar{f}_2^k(x) \\ L_{f_1}\phi_k(x) \end{bmatrix} + R_{k+1}(x) \begin{bmatrix} \bar{g}_2^k(x) \\ L_{g_1}\phi_k(x) \end{bmatrix} \alpha(x) \\ &= \begin{bmatrix} \bar{f}_2^{k+1}(x) + \bar{g}_2^{k+1}(x)\alpha(x) \\ \tilde{f}_2^{k+1}(x) \end{bmatrix}\end{aligned}$$

which implies $\tilde{f}_2^{k+1}(x) \rightarrow \tilde{f}_2^{k+1}(x)$, thus $t_{k+1} \rightarrow t_{k+1}$ and $\phi_{k+1}(x) \rightarrow \phi_{k+1}(x)$. Now the proof is completed. $\square$

Theorem 1 *Assume that x_0 is a regular point of Algorithm 1. Then, the system (1) is regularizable at x_0 if and only if $q_{k^*+1} = s$.*

Proof: **The If Part**: It follows from $q_{k^*+1} = s$ that the matrix $[\bar{p}_2^{k^*+1}(x) \quad \bar{g}_2^{k^*+1}(x)]$ has full row rank, which means that there exists a matrix $\gamma(x)$ such that $\bar{p}_2^{k^*+1}(x) + \bar{g}_2^{k^*+1}(x)\gamma(x)$ is nonsingular. Imposing the feedback $u = v + \gamma(x)z$ on the system (1), it follows from Lemma 1 that by using Algorithm 1 the corresponding closed-loop system (1) can be equivalently changed into

$$\begin{aligned} \dot{x} &= f_1(x) + [p_1(x) + g_1(x)\gamma(x)]z + g_1(x)v \\ 0 &= \bar{f}_2^{k^*+1}(x) + [\bar{p}_2^{k^*+1}(x) + \bar{g}_2^{k^*+1}(x)\gamma(x)]z + \bar{g}_2^{k^*+1}(x)v \end{aligned} \tag{10}$$

with $x_0 \in M_{k^*+1}$. It is obvious that the system (10) has a unique solution, i.e. it is regular.

The Only If Part: Assume that $q_{k^*+1} < s$. Then performing Algorithm 1 produces the following system

$$\begin{aligned} \dot{x} &= f_1(x) + p_1(x)z + g_1(x)v \\ 0 &= \bar{f}_2^{k^*+1}(x) + \bar{p}_2^{k^*+1}(x)z + \bar{g}_2^{k^*+1}(x)v \end{aligned}$$

where $[p_2^{k^*+1}(x) \quad g_2^{k^*+1}(x)]$ has full row rank $q_{k^*+1} < s$, $x \in M_{k^*+1}$. According to Lemma 1, it follows that the rank of $\bar{p}_2^{k^*+1}(x)$ can be increased to at most $q_{k^*+1} < s$ by using the feedback. As a result, the number of constrained equations is always less than the number of the variable z no matter what feedback is imposed. So, z can never be determined uniquely, i.e. the system is not regularizable at x_0. $\Box$

It is worth noting that for any regular point x_0 of Algorithm 1, a regularizable system can be equivalently described as

$$\begin{aligned} \dot{x} &= f_1(x) + p_1(x)z + g_1(x)v \\ 0 &= \bar{f}_2^{k^*+1}(x) + \bar{p}_2^{k^*+1}(x)z + \bar{g}_2^{k^*+1}(x)v \end{aligned} \tag{11}$$

where $[p_2^{k^*+1}(x) \quad g_2^{k^*+1}(x)]$ has full row rank which equals the number of the algebraic variable z, $x \in M_{k^*+1}$.

4 Noninteracting Control Problem

It follows from the last section that it is convenient to make the following assumption.

Assumption 1 *The matrix $[p_2(x) \quad g_2(x)]$ has the full row rank, say s, at x_0.*

Before discussing the noninteracting control problem, let us describe the following algorithm for the system (1) with outputs $y_i = h_i(x)$, $i = 1, \dots, m$.

Algorithm 2
Step 1. *Set* $\phi_i^0(x) = h_i(x)$ *and calculate* $L_{p_1}\phi_i^0(x), L_{g_1}\phi_i^0(x)$ *and* $L_{f_1}\phi_i^0(x)$. *If the matrix*

$$\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^0(x) & L_{g_1}\phi_i^0(x) \end{bmatrix}$$

has constant rank s *in a neighborhood of* x_0, *then by Assumption 1, there exists a unique vector-valued analytic function* $E_i^0(x)$ *of dimension* s *such that*

$$[\,L_{p_1}\phi_i^0(x) \quad L_{g_1}\phi_i^0(x)\,] = E_i^0(x)[\,p_2(x) \quad g_2(x)\,]$$

Denote $\phi_i^1(x) = L_{f_1}\phi_i^0(x) - E_i^0(x)f_2(x)$. *Otherwise, set* $r_i = 1$ *and quit the algorithm.*

Step k+1. *Assume that we have defined a sequence of* $\phi_i^0(x), \ldots, \phi_i^k(x)$. *Now calculate* $L_{p_1}\phi_i^k(x), L_{g_1}\phi_i^k(x)$ *and* $L_{f_1}\phi_i^k(x)$. *If the matrix*

$$\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^k(x) & L_{g_1}\phi_i^k(x) \end{bmatrix}$$

has constant rank s *in a neighborhood of* x_0, *then there exists a unique vector-valued analytic function* $E_i^k(x)$ *of dimension* s *such that*

$$[\,L_{p_1}\phi_i^k(x) \quad L_{g_1}\phi_i^k(x)\,] = E_i^k(x)[\,p_2(x) \quad g_2(x)\,] \tag{12}$$

Denote $\phi_i^{k+1}(x) = L_{f_1}\phi_i^k(x) - E_i^k(x)f_2(x)$. *Otherwise, set* $r_i = k+1$ *and quit the algorithm.*

Remark 3 *It can be easily seen that if* $s = 0$, *i.e. the system (1) becomes a traditional nonlinear system, then the integer* r_i *is equal to the characteristic number introduced in [22]. So, we may say that Algorithm 2 is the generalization of the definition of characteristic number to the singular nonlinear systems.*

Performing Algorithm 2 for $i = 1, \ldots, m$, we can obtain integers $r_1, \ldots, r_m$. Now, let us introduce the following matrices

$$D(x) = \begin{bmatrix} L_{f_1}\phi_1^{r_1-1}(x) \\ \cdots \\ L_{f_1}\phi_m^{r_m-1}(x) \end{bmatrix} \qquad D_z(x) = \begin{bmatrix} L_{p_1}\phi_1^{r_1-1}(x) \\ \cdots \\ L_{p_1}\phi_m^{r_m-1}(x) \end{bmatrix} \qquad D_u(x) = \begin{bmatrix} L_{g_1}\phi_1^{r_1-1}(x) \\ \cdots \\ L_{g_1}\phi_m^{r_m-1}(x) \end{bmatrix}$$

Assumption 2 *The matrix*

$$\begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix}$$

is nonsingular at x_0.

Similar to the linear case, see e.g. [7], we introduce the following nonlinear restricted equivalent transformation:

Definition 7 (*Restricted Equivalent Transformation*)
(i) Multiply the algebraic equation by any nonsingular matrix $P_3(x)$;

(ii) Add the algebraic equation multiplied by any matrix $P_2(x)$ to the differential equation;

(iii) Make the coordinates transformations $x = T(\bar{x})$;

(iv) Make the coordinates transformations $z = Q_2(x) + Q_3(x)\bar{z}$.

with the Jacobian matrix of $T(x)$, say $J_T(x)$, and $Q_3(x)$ being nonsingular at x_0.

It is easily seen that the affine form of the system (1) is invariant under the restricted equivalent transformation above.

Next we will prove that the integers $r_1, \ldots, r_m$ are invariant under the restricted equivalent transformation and under the feedback of the form $u = \alpha(x)+\beta(x)\bar{u}+\gamma(x)z$.

Lemma 2 *$(r_1, \ldots, r_m)$ are invariant under the feedback of the form $u = \alpha(x)+\beta(x)\bar{u}+\gamma(x)z$ and the restricted equivalent transformation.*

Proof: First, it is obvious that Algorithm 2 does not depend on the coordinates change in the $x-$space, i.e. the transformation (iii).

Second, it is straightforward to see that a regular feedback (2), and the operation (iv) affect all matrices in Algorithm 2 only by column operation and therefore $E_i^k(x)$, $k = 0, \ldots, r_i - 2$, $i = 1, ..., m$, remain unchanged. Moreover, one can easily see that also $\phi_i^{k+1}(x)$, $k = 0, \ldots, r_i - 2$, $i = 1, ..., m$, remain unchanged. This means that the integers, $r_1, \ldots, r_m$, are invariant under a regular feedback, and the operation (iv) in Definition 7.

Then, the operation (i), in fact, multiplies the functions $f_2(x), p_2(x)$ and $g_2(x)$ on the left by $P_3(x)$, so it is obviously seen that it only changes $E_i^k(x)$ into $E_i^k(x)P_3(x)^{-1}$. As a result, it does not effect $(r_1, \ldots, r_m)$ at all.

To conclude the proof, it is sufficient to verify that $(r_1, \ldots, r_m)$ are invariant under the operation (ii). To this end, set

$$\bar{f}_1(x) = f_1(x) + P_2(x)f_2(x)$$

$$\bar{p}_1(x) = p_1(x) + P_2(x)p_2(x)$$

$$\bar{g}_1(x) = g_1(x) + P_2(x)g_2(x)$$

and $\bar{\phi}_i^0(x) = \phi_i^0(x)$ for $i = 1, \ldots, m$. Then, it follows that

$$L_{\bar{f}_1}\bar{\phi}_i^0(x) = \mathrm{d}\phi_i^0(x)[f_1(x) + P_2(x)f_2(x)] = L_{f_1}\phi_i^0(x) + \mathrm{d}\phi_i^0(x)P_2(x)f_2(x)$$

$$L_{\bar{p}_1}\bar{\phi}_i^0(x) = \mathrm{d}\phi_i^0(x)[p_1(x) + P_2(x)p_2(x)] = L_{p_1}\phi_i^0(x) + \mathrm{d}\phi_i^0(x)P_2(x)p_2(x)$$

$$L_{\bar{g}_1}\bar{\phi}_i^0(x) = \mathrm{d}\phi_i^0(x)[g_1(x) + P_2(x)g_2(x)] = L_{g_1}\phi_i^0(x) + \mathrm{d}\phi_i^0(x)P_2(x)g_2(x)$$

from which it is easily concluded that $\bar{E}_i^0(x) = E_i^0(x) + \mathrm{d}\phi_i^0(x)P_2(x)$ and $\bar{\phi}_i^0(x) = \phi_i^0(x)$. Similarly it is easily shown that for $k = 0, \ldots, r_i - 2$ $\bar{E}_i^k(x) = E_i^k(x) + \mathrm{d}\phi_i^k(x)P_2(x)$ and $\bar{\phi}_i^{k+1}(x) = \phi_i^{k+1}(x)$.

In addition, because

$$L_{\bar{p}_1}\bar{\phi}_i^{r_i-1}(x) = \mathrm{d}\phi_i^{r_i-1}(x)[p_1(x) + P_2(x)p_2(x)] = L_{p_1}\phi_i^{r_i-1}(x) + \mathrm{d}\phi_i^{r_i-1}(x)P_2(x)p_2(x)$$

$$L_{\bar{g}_1}\bar{\phi}_i^{r_i-1}(x) = \mathrm{d}\phi_i^{r_i-1}(x)[g_1(x) + P_2(x)g_2(x)] = L_{g_1}\phi_i^{r_i-1}(x) + \mathrm{d}\phi_i^{r_i-1}(x)P_2(x)g_2(x)$$

for $i = 1, \ldots, m$,

$$\begin{aligned}
& \operatorname{rank}\begin{bmatrix} \bar{p}_2(\bar{x}) & \bar{g}_2(\bar{x}) \\ L_{\bar{p}_1}\bar{\phi}_i^{r_i-1}(\bar{x}) & L_{\bar{g}_1}\bar{\phi}_i^{r_i-1}(\bar{x}) \end{bmatrix} \\
= & \operatorname{rank}\left\{\begin{bmatrix} I & 0 \\ \mathrm{d}\phi_i^{r_i-1}(x)P_2(x) & I \end{bmatrix}\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix}\right\} \\
= & \operatorname{rank}\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix} \\
= & s+1
\end{aligned}$$

So the integers $r_1, \ldots, r_m$ are invariant under the operation (ii). This completes the proof of the lemma. □

Next, we can prove the following conclusions.

Lemma 3 *Suppose that Assumptions 1 and 2 hold. Then the vectors*

$$\mathrm{d}\phi_1^0, \ldots, \mathrm{d}\phi_1^{r_1-1}, \ldots, \mathrm{d}\phi_m^0, \ldots, \mathrm{d}\phi_m^{r_m-1}$$

are linearly independent at the point x_0.

Proof: Assume that

$$\mathrm{d}\phi_1^0, \ldots, \mathrm{d}\phi_1^{r_1-1}, \ldots, \mathrm{d}\phi_m^0, \ldots, \mathrm{d}\phi_m^{r_m-1}$$

are not linearly independent at the point x_0. Then

$$0 = \sum_{i=1}^{m}\sum_{t=0}^{r_i-2} c_i^t \mathrm{d}\phi_i^t(x_0) + \sum_{i=1}^{m} c_i^{r_i-1}\mathrm{d}\phi_i^{r_i-1}t(x_0)$$

with $c_i^t, t = 0, \ldots, r_i - 1, i = 1, \ldots, m$, being constant. Consequently, one can derive

$$\begin{aligned}
0 &= \{\textstyle\sum_{i=1}^m \sum_{t=0}^{r_i-2} c_i^t \mathrm{d}\phi_i^t(x_0) + \sum_{i=1}^m c_i^{r_i-1}\mathrm{d}\phi_i^{r_i-1}(x_0)\}\,[\,p_1(x_0) \quad g_1(x_0)\,] \\
&= \textstyle\sum_{i=1}^m \sum_{t=0}^{r_i-2} c_i^t\,[\,\mathrm{d}\phi_i^t(x_0)p_1(x_0) \quad \mathrm{d}\phi_i^t(x_0)g_1(x_0)\,] \\
&\quad + \textstyle\sum_{i=1}^m c_i^{r_i-1}\,[\,\mathrm{d}\phi_i^{r_i-1}(x_0)p_1(x_0) \quad \mathrm{d}\phi_i^{r_i-1}(x_0)g_1(x_0)\,] \\
&= \textstyle\sum_{i=1}^m \sum_{t=0}^{r_i-2} c_i^t\,[\,L_{p_1}\phi_i^t(x_0) \quad L_{g_1}\phi_i^t(x_0)\,] + \sum_{i=1}^m c_i^{r_i-1}\,[\,L_{p_1}\phi_i^{r_i-1}(x_0) \quad L_{g_1}\phi_i^{r_i-1}(x_0)\,]
\end{aligned}$$

which implies that the matrix

$$\begin{bmatrix} p_2(x_0) & g_2(x_0) \\ D_z(x_0) & D_u(x_0) \end{bmatrix} = \begin{bmatrix} p_2(x_0) & g_2(x_0) \\ L_{p_1}\phi_1^{r_1-1}(x_0) & L_{g_1}\phi_1^{r_1-1}(x_0) \\ \vdots & \vdots \\ L_{p_m}\phi_m^{r_m-1}(x_0) & L_{g_m}\phi_m^{r_m-1}(x_0) \end{bmatrix}$$

is singular. This contradicts to Assumption 2, i.e. the lemma has been proved. □

It follows from Lemma 3 that it is always possible to select $(n-r)$ smooth functions, say $\psi_1, \ldots, \psi_{(n-r)}$, such that the vectors

$$\mathrm{d}\phi_1^0, \ldots, \mathrm{d}\phi_1^{r_1-1}, \ldots, \mathrm{d}\phi_m^0, \ldots, \mathrm{d}\phi_m^{r_m-1}\mathrm{d}\psi_1, \ldots, \mathrm{d}\psi_{(n-r)}$$

are linearly independent at x_0, that is,

$$\mathrm{col}(\phi_1^0, \ldots, \phi_1^{r_1-1}, \ldots, \phi_m^0, \ldots, \phi_m^{r_m-1}\psi_1, \ldots, \psi_{(n-r)})$$

constitutes a local coordinates transformation, where $r = r_1 + \cdots + r_m$. Now let

$$\begin{aligned} \xi_i^k &= \phi_i^{k-1}(x), \quad k = 1, \ldots, r_i, i = 1, \ldots, m \\ \eta_i &= \psi_i(x), \qquad\quad i = 1, \ldots, n-r \\ \begin{bmatrix} \eta \\ \xi \end{bmatrix} &= \phi(x) \end{aligned}$$

Then it is easily derived that the description of the system (1) in the new coordinates admits the following form

$$\begin{aligned} \dot{\eta} &= \bar{f}_1(\eta, \xi) + \bar{p}_1(\eta, \xi)z + \bar{g}_1(\eta, \xi)u \\ 0 &= [f_2(x) + p_2(x)z + g_2(x)u]_{x=\phi^{-1}(\eta,\xi)} \\ \dot{\xi}_i^1 &= \xi_i^2 \\ &\vdots \\ \dot{\xi}_i^{r_i-1} &= \xi_i^{r_i} \\ \dot{\xi}^{r_i} &= [L_{f_1}\phi_i^{r_i-1}(x) + L_{p_1}\phi_i^{r_i-1}(x)z + L_{g_1}\phi_i^{r_i-1}(x)u]_{x=\phi^{-1}(\eta,\xi)} \\ y_i &= \xi_i^1 \end{aligned}$$

$$i = 1, \ldots, m \tag{13}$$

Now the following feedback law is imposed

$$\begin{aligned} u = & -\{D_u(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\}^{-1} \times \\ & \{D(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\ & +\{D_u(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\}^{-1}v + \gamma(x)z \end{aligned}$$

with $p_2(x) + g_2(x)\gamma(x)$ nonsingular at x_0. An immediate calculation shows that the corresponding closed-loop system becomes as follows:

$$\begin{aligned} \dot{\eta} &= \hat{f}_1(\eta, \xi) + \hat{p}_1(\eta, \xi)z + \hat{g}_1(\eta, \xi)v \\ 0 &= \hat{f}_2(\eta, \xi) + [p_2(x) + g_2(x)\gamma(x)]_{x=\phi^{-1}(\eta,\xi)}z + \hat{g}_2(\eta, \xi)v \\ \dot{\xi}_i^1 &= \xi_i^2 \\ &\vdots \\ \dot{\xi}_i^{r_i-1} &= \xi_i^{r_i} \\ \dot{\xi}^{r_i} &= v_i \\ y_i &= \xi_i^1 \end{aligned}$$

$$i = 1, \ldots, m \tag{14}$$

The structure of these equations shows that the noninteraction requirements have been achieved. As a mater of fact, the input v_1 controls only the output y_1, throughout a chain of r_1 integrators, the input v_2 controls only the output y_2, throughout a chain of r_2 integrators, etc. In addition, the nonsingularity of the matrix $p_2(x) + g_2(x)\gamma(x)$ guarantees that the system (14) has the property (a) given in Definition 5. Therefore the assumption 2 is a sufficient condition for the solvability of the noninteraction control problem. In fact, it is also necessary.

Theorem 2 *Suppose that Assumption 1 holds and the matrix*

$$\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix}$$

is of full row rank at x_0 for all $i = 1, \ldots, m$. Then the noninteracting control problem is solvable if and only if Assumption 2 is satisfied.

Proof: Sufficiency was already verified. Now let us prove its necessity. To this end, assume that the closed-loop system (3) has been rendered noninteractive, i.e. possesses the properties (a) and (b). It is easily seen that a necessary condition under which the system (3) has the property (a) is that the matrix $p_2(x) + g_2(x)\gamma(x)$ is nonsingular at x_0. As a result, z can be uniquely determined from the second equation of (3) as

$$z = -[p_2(x) + g_2(x)\gamma(x)]^{-1}[f_2(x) + g_2(x)\alpha(x) + g_2(x)\beta(x)v]$$

Substituting this into the first equation of (3) yields

$$\begin{aligned} \dot{x} &= \{f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\ &\quad +\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\}\alpha(x) \\ &\quad +\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\}\beta(x)v \\ y &= h(x) \end{aligned} \tag{15}$$

According to the noninteractive property of the system (3), it follows that the system (15) has the property (b), that is, the system

$$\begin{aligned} \dot{x} &= \{f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\ &\quad +\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\}w \\ y &= h(x) \end{aligned} \tag{16}$$

can be decoupled by the feedback $w = \alpha(x) + \beta(x)v$.

Next we use that following two lemmas.

Lemma 4 *Suppose that Assumption 1 holds and the matrix*

$$\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix}$$

is of full row rank at x_0 for all $i = 1, \ldots, m$. Then for any matrix $\gamma(x)$ which renders $p_2(x) + g_2(x)\gamma(x)$ nonsingular at x_0,

$$L_g L_f^k h_i(x) = 0, \qquad k = 0, \ldots, r_i - 2, i = 1, \ldots, m$$

for all x in a neighborhood of x_0, and

$$L_g L_f^{r_i-1} h_i(x_0) \neq 0, \qquad i = 1, \ldots, m$$

with

$$\begin{aligned} f(x) &= f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1} f_2(x) \\ g(x) &= g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1} g_2(x) \end{aligned}$$

Proof: See Appendix.

Lemma 5 *Suppose that Assumption 1 holds. Then the system (16) has a vector relative degree $(r_1, \ldots, r_m)$ if and only if Assumption 2 is satisfied.*

Proof: See Appendix.

According to the results about the noninteracting control problem of the regular system [8], it follows from Lemma 4 that the system (16) has a vector relative degree $(r_1, \ldots, r_m)$. According to Lemma 5, one can derive that Assumption 2 is necessary for the system (3) to have the property (b). This completes the proof of the theorem. □

It is important to note that for any matrix $\gamma(x)$ which renders $p_2(x) + g_2(x)\gamma(x)$ nonsingular, the reletive degree of the system (16) coincides with a set of the integers $r_1, \ldots, r_m$ defined by Algorithm 2 for the system (1) with outputs $y_i = h_i(x), i = 1, \ldots, m$. Therefore, it is convenient to call such a set of integers as a generalized relative degree of the system (1).

5 Dynamic Equation Exact Linearization Problem

This section will investigate the dynamic equation exact linearization problem. According to the discussion given in the last section, it is obvious that if $r = n$, then the closed-loop system (14) becomes as follows

$$\begin{aligned} \dot{\xi}_i^1 &= \xi_i^2 \\ &\vdots \\ \dot{\xi}_i^{r_i-1} &= \xi_i^{r_i} \\ \dot{\xi}^{r_i} &= v_i \\ y_i &= \xi_i^1 \\ 0 &= \hat{f}_2(\eta, \xi) + [p_2(x) + g_2(x)\gamma(x)]_{x=\phi^{-1}(\eta,\xi)} z + \hat{g}_2(\eta, \xi) v \\ & \qquad\qquad i = 1, \ldots, m \end{aligned} \tag{17}$$

which possesses the form of (4).

Thus, it follows that the sufficient condition for the solvability of the Dynamic Equation Exact Linearization Problem is that there exist $m-$smooth functions $h_1(x), \ldots, h_m(x)$ such that the system with outputs $y_i = h_i(x), i = 1, \ldots, m$, has a generalized relative degree $\{r_1, \ldots, r_m\}$ at x_0 and $r_1 + \cdots + r_m = n$. We show now that the condition in question is also necessary.

Lemma 6 *Suppose that Assumption 2 holds. Then the Dynamic Equation Exact Linearization Problem is solvable if and only if there exist* $m-$*smooth functions* $h_1(x), \ldots, h_m(x)$ *such that the system with outputs* $y_i = h_i(x), i = 1, \ldots, m$*, has a generalized relative degree* $\{r_1, \ldots, r_m\}$ *at* x_0 *and* $r_1 + \cdots + r_m = n$*.*

Proof. Clearly, we only have to show that the conditions are necessary. Without loss of generality, we may assume that the matrices A and B in (4) have the form (Brunowsky canonical form)

$$a = diag(A_1, \ldots, A_m) \qquad B = diag(b_1, \ldots, b_m)$$

where A_i is the $r_i \times r_i$ matrix

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and b_i is the $r_i \times 1$ vector

$$b_i = col(0, \ldots, 0, 1)$$

Now decompose ξ into $\xi = col(\xi^1, \ldots, \xi^m)$ and set

$$y_i = [1 \;\; 0 \;\; \cdots \;\; 0]\xi^i, \qquad i = 1, \ldots, m$$

with $\dim(\xi^i) = r_i$, $i = 1, \ldots, m$. A straightforward calculation shows that the system

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ 0 &= f_2(\xi) + p_2(\xi)z + g_2(\xi)v \end{aligned}$$

with output functions $y_i = [1 \;\; 0 \;\; \cdots \;\; 0]\xi^i, \qquad i = 1, \ldots, m$, has a generalized relative degree $\{r_1, \ldots, r_m\}$. It follows from Lemma 2 that the condition is necessary. $\square$

In order to find m smooth functions $h_1(x), \ldots, h_m(x)$ satisfying the requirements of Lemma 6, we provide the following lemma.

Lemma 7 *Suppose that Assumption 2 holds. Then the following statements are equilvalent*

1. *there exist* $m-$*smooth functions* $h_1(x), \ldots, h_m(x)$ *such that the system with outputs* $y_i = h_i(x), i = 1, \ldots, m$*, has a generalized relative degree* $\{r_1, \ldots, r_m\}$ *at* x_0 *and* $r_1 + \cdots + r_m = n$*;*

2. *there exist* $m-$*smooth functions* $h_1(x), \ldots, h_m(x)$ *such that for any matrix* $\gamma(x)$ *which renders* $p_2(x) + g_2(x)\gamma(x)$ *invertible, the system*

$$\dot{x} = f(x) + g(x)v \tag{18}$$

with outputs $y_i = h_i(x), i = 1, \ldots, m$*, has a relative degree* $\{r_1, \ldots, r_m\}$ *at* x_0 *and* $r_1 + \cdots + r_m = n$ *where*

$$f(x) = f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)$$

$$g(x) = g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)$$

3. *For any $\gamma(x)$ which makes $p_2(x) + g_2(x)\gamma(x)$ invertible, the system (18) is linearizable at around x_0, that is, the distributions $D_1, \ldots, D_n$ are all involutive and nonsingular at x_0 where*

$$D_k(x) = span\{ad_f^r g(x), r = 0, \ldots, k-1\}, \qquad k = 0, 1, \ldots$$

Proof. The equivelence of 1 and 2 is easily proved according to the last section. According to [8], 2 is equivalent to 3. $\square$

Now in view of the previous discussion, we can state the necessary and sufficient conditions for the solvability of the Dynamic Equation Exact Linearization Problem in the following way.

Theorem 3 *Suppose that Assumption 2 holds. Then the Dynamic Equation Exact Linearization Problem is solvable if and only if for any any $\gamma(x)$ which makes $p_2(x) + g_2(x)\gamma(x)$ invertible, the distributions $D_1, \ldots, D_n$ are all involutive and nonsingular at x_0.*

6 Example

In this section, we consider the following singular nonlinear system

$$\begin{aligned}
\dot{\psi} &= \begin{bmatrix} \psi_2 + (\psi_3)^2 + \psi_2(\psi_4)^2 e^{\psi_3} \\ \psi_1\psi_4 - 2\psi_2(\psi_3)^2 \\ \psi_2\psi_3 \\ -(\psi_4)^3 + \psi_1[\psi_2 + (\psi_3)^2] \\ -2\psi_1\psi_2 + \psi_2(\psi_4)^2(1 - 2\psi_1 e^{\psi_3}) - 2\psi_1(\psi_3)^2 + \psi_6 + (\psi_5)^2 \\ \psi_1\psi_3\psi_4 - 2\psi_5\{-2\psi_1\psi_2 + \psi_2(\psi_4)^2(1 - 2\psi_1 e^{\psi_3}) - 2\psi_1(\psi_3)^2 + \psi_6 + (\psi_5)^2\} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \psi_2\psi_3 & 1 \\ \psi_1 & \psi_3 \\ 0 & 0 \\ \psi_2\psi_3 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & e^{\psi_3} \\ e^{\psi_3} - 2\psi_3 & -2\psi_3 e^{\psi_4} \\ 1 + \psi_3 e^{\psi_3} & e^{\psi_4} - 2 \\ \psi_3 e^{\psi_3} & 0 \\ 0 & 1 - 2\psi_1 e^{\psi_3} \\ \psi_3 e^{\psi_3} & -2\psi_5(1 - 2\psi_1 e^{\psi_3}) \end{bmatrix} u \\
0 &= \begin{bmatrix} \psi_2(\psi_4)^2 \\ (\psi_1)^2 + \psi_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u \\
y_1 &= \psi_1 \\
y_2 &= \psi_3
\end{aligned}$$

Evidently the system does not satisfy Assumption 1. However, by using Algorithm 1, it follows that under the coordinates transformation

$$\begin{aligned}
x_1 &= \psi_1 \\
x_2 &= \psi_2 \\
x_3 &= \psi_3 \\
x_4 &= \psi_4 \\
x_5 &= (\psi_1)^2 + \psi_5 \\
x_6 &= \psi_6 + (\psi_5)^2
\end{aligned}$$

the original system becomes

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} x_2 + (x_3)^2 + x_2(x_4)^2 \mathrm{e}^{x_3} \\ x_1 x_4 - 2x_2(x_3)^2 \\ x_2 x_3 \\ -(x_4)^3 + x_1[x_2 + (x_3)^2] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ x_2 x_3 & 1 \\ x_1 & x_3 \end{bmatrix} z + \begin{bmatrix} 0 & \mathrm{e}^{x_3} \\ \mathrm{e}^{x_3} - 2x_3 & -2x_3 \mathrm{e}^{x_4} \\ 1 + x_3 \mathrm{e}^{x_3} & \mathrm{e}^{x_4} - 2 \\ x_3 \mathrm{e}^{x_3} & 0 \end{bmatrix} u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= x_1 x_3 x_4 + x_2 x_3 z_1 + z_2 + x_3 \mathrm{e}^{x_3} u_1 \\
0 &= \begin{bmatrix} x_2(x_4)^2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned}
\tag{19}
$$

So for any initial state x_0 which belongs to the manifold $M = \{x \in R^6 : x_5 = x_6 = 0\}$, the system (19) is equivalent to the following one

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} x_2 + (x_3)^2 + x_2(x_4)^2 \mathrm{e}^{x_3} \\ x_1 x_4 - 2x_2(x_3)^2 \\ x_2 x_3 \\ -(x_4)^3 + x_1[x_2 + (x_3)^2] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ x_2 x_3 & 1 \\ x_1 & x_3 \end{bmatrix} z + \begin{bmatrix} 0 & \mathrm{e}^{x_3} \\ \mathrm{e}^{x_3} - 2x_3 & -2x_3 \mathrm{e}^{x_4} \\ 1 + x_3 \mathrm{e}^{x_3} & \mathrm{e}^{x_4} - 2 \\ x_3 \mathrm{e}^{x_3} & 0 \end{bmatrix} u \\
0 &= \begin{bmatrix} x_2(x_4)^2 \\ x_1 x_3 x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_2 x_3 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ x_3 \mathrm{e}^{x_3} & 0 \end{bmatrix} u \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned}
\tag{20}
$$

It is easily seen that this system satisfies Assumption 1. In order to check whether or not the system can be decoupled, we may perform Algorithm 2 and test Assumption 2. Appropriate calculations show that

$$
\begin{aligned}
\phi_1^0(x) &= x_1 \\
E_1^0(x) &= [\mathrm{e}^{x_3} \quad 0] \\
\phi_1^1(x) &= x_2 + (x_3)^2 \\
\phi_2^0(x) &= x_3 \\
D_x(x) &= \begin{bmatrix} x_1 x_4 \\ x_2 x_3 \end{bmatrix} \\
D_z(x) &= \begin{bmatrix} 1 + 2x_2(x_3)^2 & 2x_3 \\ x_2 x_3 & 1 \end{bmatrix} \\
D_u(x) &= \begin{bmatrix} [2(x_3)^2 + 1]\mathrm{e}^{x_3} & -4x_3 \\ 1 + x_3 \mathrm{e}^{x_3} & \mathrm{e}^{x_4} - 2 \end{bmatrix}
\end{aligned}
$$

It is easy to check that Assumption 2 is satisfied. So the original system can be decoupled. In fact, under the coordinates transformation

$$
\begin{aligned}
\xi_1 &= x_1 \\
\xi_2 &= x_2 + (x_3)^2 \\
\xi_3 &= x_3 \\
\eta &= x_4
\end{aligned}
$$

the original system becomes as follows:

$$\begin{aligned}
y_1 &= \xi_1\\
\dot{\xi}_1 &= \xi_2\\
\dot{\xi}_2 &= x_1x_4 + [1+2x_2(x_3)^2]z_1 + 2x_3z_2 + [2(x_3)^2+1]\mathrm{e}^{x_3}u_1 - 4x_3u_2\\
y_2 &= \xi_3\\
\dot{\xi}_3 &= x_2x_3 + x_2x_3z_1 + z_2 + (1+x_3\mathrm{e}^{x_3})u_1 + (\mathrm{e}^{x_4}-2)u_2\\
\dot{\eta} &= -\eta^3 + \xi_1\xi_2 + \xi_1z_1 + \xi_3z_2 + \xi_3\mathrm{e}^{\xi_3}u_1\\
0 &= \begin{bmatrix} x_2(x_4)^2 \\ x_1x_3x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_2x_3 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ x_3\mathrm{e}^{x_3} & 0 \end{bmatrix} u
\end{aligned}$$

In addition, by using the feedback

$$\begin{aligned}
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} x_1x_3x_4 - x_2[x_3+(x_4)^2(2-\mathrm{e}^{x_4})] \\ x_1x_4[1-2(x_3)^2] - x_2(x_4)^2(1-4x_3) + \{x_1x_3x_4 - x_2[x_3+(x_4)^2(2-\mathrm{e}^{x_4})]\}\mathrm{e}^{x_3} \end{bmatrix}\\
&\quad + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} v
\end{aligned}$$

the closed-loop system takes the form of

$$\begin{aligned}
y_1 &= \xi_1\\
\dot{\xi}_1 &= \xi_2\\
\dot{\xi}_2 &= v_1\\
y_2 &= \xi_3\\
\dot{\xi}_3 &= v_2\\
\dot{\eta} &= -\eta^3 + \xi_1\xi_2 + \xi_3\{\xi_1\xi_3\eta - [\xi_2-(\xi_3)^2][\xi_3+\eta^2(2-\mathrm{e}^{\eta})]\}\mathrm{e}^{\xi_3} + \xi_1z_1 + \xi_3z_2 + \xi_3\mathrm{e}^{\xi_3}v_2\\
0 &= \begin{bmatrix} \xi_1\eta[1-2(\xi_3)^2] + 4[\xi_2-(\xi_3)^2]\xi_3\eta^2 + \{\xi_1\xi_3\eta - [\xi_2-(\xi_3)^2][\xi_3+\eta^2(2-\mathrm{e}^{\eta})]\}\mathrm{e}^{\xi_3} \\ \xi_1\xi_3\eta + \xi_3\{\xi_1\xi_3\eta - [\xi_2-(\xi_3)^2][\xi_3+\eta^2(2-\mathrm{e}^{\eta})]\}\mathrm{e}^{\xi_3} \end{bmatrix}\\
&\quad + \begin{bmatrix} 1 & 0 \\ [\xi_2-(\xi_3)^2]\xi_3 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 \\ 0 & \xi_3\mathrm{e}^{\xi_3} \end{bmatrix} v
\end{aligned} \tag{21}$$

which possesses the properties (a) and (b) in Definition 5.

It is worth noting that this system can be decoupled with stability. In fact, solving z from the algebraic equation of (21) and substituting it into the 6th equation of (21), then letting $\xi_1 = \xi_2 = \xi_3 = 0$, it follows that the zero-dynamic of the system becomes $\dot{\eta} = -\eta^3$, which is asymptotically stable at $\eta = 0$. Therefore the noninteracting control problem with stability is solvable for the system.

7 Conclusion

The regularization problem has been discussed for affine nonlinear singular systems. By using an algorithm similar to the zero-dynamic algorithm or constrained dynamic algorithm, necessary and sufficient conditions for the solvability of the regularization problem has been derived. Based on this result, the noninteracting control problem has been investigated in detail for the same kind of systems. A new algorithm has been proposed, which provides a condition for existence as well as a procedure for construction of a decoupling feedback law. By using the results on the noninteracting control problem, the condition for the solvability of dynamic equation exact linearization problem has been given.

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Appendix

1. Proof of Lemma 4

It easily follows from Algorithm 2 that there exists a neighborhood U of x_0 such that the relation

$$[\, L_{p_1}\phi_i^k(x) \quad L_{g_1}\phi_i^k(x)\,] = E_i^k(x)[\, p_2(x) \quad g_2(x)\,] \tag{22}$$

holds for $k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U$. It is well-known that the equation (22) is equivalent to the following equation

$$[\, L_{p_1}\phi_i^k(x) \quad L_{g_1}\phi_i^k(x)\,] \begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix} = E_i^k(x)[\, p_2(x) \quad g_2(x)\,] \begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix}$$

that is

$$[\, L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x) \quad L_{g_1}\phi_i^k(x)\,] = E_i^k(x)[\, p_2(x) + g_2(x)\gamma(x) \quad g_2(x)\,]$$

for $k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U$. As a result, for any $\gamma(x)$ such that $[p_2(x)+g_2(x)\gamma(x)]$ is nonsingular, $E_i^k(x)$ can be uniquely determined as

$$E_i^k(x) = [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1} \tag{23}$$

with $k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U$. Substituting $E_i^k(x)$ into (22) gives the following relations:

$$\begin{aligned} L_{p_1}\phi_i^k(x) &= [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}p_2(x) \\ L_{g_1}\phi_i^k(x) &= [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \end{aligned}$$

$$k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U \tag{24}$$

By the construction of $\phi_i^{k+1}(x)$, it is easily seen that

$$\begin{aligned} \phi_i^{k+1}(x) &= L_{f_1}\phi_i^k(x) - E_i^k(x)f_2(x) \\ &= L_{f_1}\phi_i^k(x) - [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x) \end{aligned}$$

$$k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U \tag{25}$$

In addition, since

$$\begin{aligned} & \operatorname{rank}\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix} \\ = \; & \operatorname{rank}\left\{\begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^{r_i-1}(x) & L_{g_1}\phi_i^{r_i-1}(x) \end{bmatrix}\begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix}\begin{bmatrix} I & -[p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\ 0 & I \end{bmatrix}\right\} \\ = \; & \operatorname{rank}\begin{bmatrix} p_2(x)+g_2(x)\gamma(x) & 0 \\ X & L_{g_1}\phi_i^{r_i-1}(x) - [L_{p_1}\phi_i^{r_i-1}(x) + L_{g_1}\phi_i^{r_i-1}(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \end{bmatrix} \end{aligned}$$

is of full row rank at x_0, it follows that for $i = 1, \ldots, m$

$$L_{g_1}\phi_i^{r_i-1}(x_0) - [L_{p_1}\phi_i^{r_i-1}(x_0) + L_{g_1}\phi_i^{r_i-1}(x_0)\gamma(x_0)][p_2(x_0) + g_2(x_0)\gamma(x_0)]^{-1}g_2(x_0) \neq 0 \tag{26}$$

According to (22)-(26), it follows that for $i = 1, \ldots, m$ and $x \in U$

$$\begin{array}{rcl}
L_g h_i(x) & = & dh_i(x)g(x) \\
& = & \mathrm{d}\phi_i^0(x)\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\} \\
& = & L_{g_1}\phi_i^0(x) - [L_{p_1}\phi_i^0(x) + L_{g_1}\phi_i^0(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \\
& = & 0 \\
L_f h_i(x) & = & dh_i(x)f(x) \\
& = & \mathrm{d}\phi_i^0(x)\{f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\
& = & L_{f_1}\phi_i^0(x) - [L_{p_1}\phi_i^0(x) + L_{g_1}\phi_i^0(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x) \\
& = & \phi_i^1(x) \\
L_g L_f h_i(x) & = & \mathrm{d}\phi_i^1(x)\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\} \\
& = & L_{g_1}\phi_i^1(x) - [L_{p_1}\phi_i^1(x) + L_{g_1}\phi_i^1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \\
& = & 0 \\
L_f h_i(x) & = & \mathrm{d}\phi_i^1(x)\{f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\
& = & L_{f_1}\phi_i^1(x) - [L_{p_1}\phi_i^1(x) + L_{g_1}\phi_i^1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x) \\
& = & \phi_i^2(x) \\
& \vdots & \\
L_g L_f^{r_i-2} h_i(x) & = & \mathrm{d}\phi_i^{r_i-2}(x)\{g_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x)\} \\
& = & L_{g_1}\phi_i^{r_i-2}(x) - [L_{p_1}\phi_i^{r_i-2}(x) + L_{g_1}\phi_i^{r_i-2}(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \\
& = & 0 \\
L_f^{r_i-1} h_i(x) & = & \mathrm{d}\phi_i^{r_i-2}(x)\{f_1(x) - [p_1(x) + g_1(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x)\} \\
& = & L_{f_1}\phi_i^{r_i-2}(x) - [L_{p_1}\phi_i^{r_i-2}(x) + L_{g_1}\phi_i^{r_i-2}(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}f_2(x) \\
& = & \phi_i^{r_i-1}(x) \\
L_g L_f^{r_i-1} h_i(x_0) & = & \mathrm{d}\phi_i^{r_i-1}(x_0)\{g_1(x_0) - [p_1(x_0) + g_1(x_0)\gamma(x_0)][p_2(x_0) + g_2(x_0)\gamma(x_0)]^{-1}g_2(x_0)\} \\
& = & L_{g_1}\phi_i^{r_i-1}(x_0) \\
& & -[L_{p_1}\phi_i^{r_i-1}(x_0) + L_{g_1}\phi_i^{r_i-1}(x_0)\gamma(x_0)][p_2(x_0) + g_2(x_0)\gamma(x_0)]^{-1}g_2(x_0) \\
& \neq 0 &
\end{array} \tag{27}$$

which completes the proof of the lemma.

2. Proof of Lemma 5

Sufficient Part:Since for any $\gamma(x)$ which renders $p_2(x) + g_2(x)\gamma(x)$ nonsingular at x_0,

$$\begin{array}{rl}
& \operatorname{rank}\begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix} \\
= & \operatorname{rank}\left\{\begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix}\begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix}\begin{bmatrix} I & -[p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \\ 0 & I \end{bmatrix}\right\} \\
= & \operatorname{rank}\begin{bmatrix} p_2(x) + g_2(x)\gamma(x) & 0 \\ [D_z(x) + D_u(x)\gamma(x)] & D_u(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x) + g_2(x)\gamma(x)]^{-1}g_2(x) \end{bmatrix}
\end{array}$$

is nonsingular at x_0, it follows from (27) that the decoupling matrix

$$
\begin{aligned}
& A(x) \\
= & \begin{bmatrix} L_{g_1}\phi_1^{r_1-1}(x) - [L_{p_1}\phi_1^{r_1-1}(x) + L_{g_1}\phi_1^{r_1-1}(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\ \vdots \\ L_{g_1}\phi_m^{r_m-1}(x) - [L_{p_1}\phi_m^{r_m-1}(x) + L_{g_1}\phi_m^{r_m-1}(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \end{bmatrix} \\
= & \begin{bmatrix} L_{g_1}\phi_1^{r_1-1}(x) \\ \vdots \\ L_{g_1}\phi_m^{r_m-1}(x) \end{bmatrix} - \left\{ \begin{bmatrix} L_{p_1}\phi_1^{r_1-1}(x) \\ \vdots \\ L_{p_1}\phi_m^{r_m-1}(x) \end{bmatrix} + \begin{bmatrix} L_{g_1}\phi_1^{r_1-1}(x) \\ \vdots \\ L_{g_1}\phi_m^{r_m-1}(x) \end{bmatrix} \gamma(x) \right\} [p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\
= & D_u(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x)
\end{aligned}
\tag{28}
$$

is nonsingular at x_0, which implies the system (16) has a vector relative degree $(r_1, \ldots, r_m)$.

Necessary Part: Assume that the system (16) has a vector relative degree $(r_1, \ldots, r_m)$. Then calculating similarly to (27) yields

$$
\begin{aligned}
L_{g_1}\phi_i^k(x) &= [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\
& \qquad k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
\phi_i^{k+1}(x) = L_{f_1}\phi_i^k(x) - [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}f_2(x) \\
k = 0, \ldots, r_i - 2, i = 1, \ldots, m, x \in U
\end{aligned}
\tag{30}
$$

Therefore,

$$
\begin{aligned}
& \operatorname{rank} \begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^k(x) & L_{g_1}\phi_i^k(x) \end{bmatrix} \\
= & \operatorname{rank} \left\{ \begin{bmatrix} p_2(x) & g_2(x) \\ L_{p_1}\phi_i^k(x) & L_{g_1}\phi_i^k(x) \end{bmatrix} \begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix} \begin{bmatrix} I & -[p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\ 0 & I \end{bmatrix} \right\} \\
= & \operatorname{rank} \begin{bmatrix} p_2(x)+g_2(x)\gamma(x) & 0 \\ L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x) & L_{g_1}\phi_i^k(x) - [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \end{bmatrix} \\
= & s
\end{aligned}
$$

and

$$
E_i^k(x) = [L_{p_1}\phi_i^k(x) + L_{g_1}\phi_i^k(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}
$$

It follows from (30) that

$$
\phi_i^{k+1}(x) = L_{f_1}\phi_i^k(x) - E_i^k(x)f_2(x)
$$

In addition,

$$
\begin{aligned}
& \operatorname{rank} \begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix} \\
= & \operatorname{rank} \left\{ \begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix} \begin{bmatrix} I & 0 \\ \gamma(x) & I \end{bmatrix} \begin{bmatrix} I & -[p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \\ 0 & I \end{bmatrix} \right\} \\
= & \operatorname{rank} \begin{bmatrix} p_2(x)+g_2(x)\gamma(x) & 0 \\ [D_z(x) + D_u(x)\gamma(x)] & D_u(x) - [D_z(x) + D_u(x)\gamma(x)][p_2(x)+g_2(x)\gamma(x)]^{-1}g_2(x) \end{bmatrix} \\
= & \operatorname{rank} \begin{bmatrix} p_2(x)+g_2(x)\gamma(x) & 0 \\ [D_z(x) + D_u(x)\gamma(x)] & A(x) \end{bmatrix}
\end{aligned}
$$

From the invertibility of $A(x)$ and $p_2(x) + g_2(x)\gamma(x)$ at x_0, it follows that the matrix $\begin{bmatrix} p_2(x) & g_2(x) \\ D_z(x) & D_u(x) \end{bmatrix}$ is nonsingular at x_0.