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J.J. Hench C. He V. Mehrmann

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ÚTIA AV ČR P. O. Box 18, 182 08 Prague, Czech Republic
Telex: 122018 atom c, Fax: (+42)(2)688 4903
e-mail: utia@utia.cas.cz

A Note on Damped Algebraic Riccati Equations*

J.J. Hench[†] C-Y. He[‡] V. Mehrmann[⋆]

[†]Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4
128 08 Praha 8
Czech Republic

[‡] Department of Mathematics
University of Kansas
Lawrence, KN 66045
USA

[⋆] Technische Universität Chemnitz-Zwickau
Fakultät für Mathematik
D-09107 Chemnitz
Germany

e-mail:

`hench@utia.cas.cz`
`chunyang@math.ukans.edu`
`mehrmann@mathematik.tu-chemnitz.de`

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Abstract

In a recent paper, an algorithm which produces dampening controllers based on a periodic formulation of Hamiltonian systems was proposed. Central to this algorithm is a formulation of symmetric and skew-symmetric damped algebraic Riccati equations. It was shown that each of the two solutions leads to a dampening feedback, *i.e.,* a stable closed–loop system for which the real parts of the eigenvalues are larger in modulus than the imaginary parts.

In this paper, we extend these results to include a broader class of Hermitian and skew-Hermitian solutions and show that every convex combination of these solutions produces a dampening feedback. This property can be used to vary the feedback with two parameters and thus obtain more flexibility in the controller design process.

Keywords: dampening feedback, damped system dynamics, periodic Schur decomposition, periodic Riccati equation, Hamiltonian systems, linear quadratic control.

1 Introduction

This paper is concerned with the problem of finding a feedback for a linear time-invariant system for which the closed–loop system poles are constrained to cones in the left half–plane. At first glance, it seems that this problem may be trivially solved by pole placement; however, this may not be desirable, especially for large systems, as pole placement has been shown to be an inherently ill–conditioned problem [1, 6]. While this problem may be satisfactorally solved in a polynomial framework [5], the formulation in a state-space framework had until just recently eluded simple formulation. The additional effort to formulate the problem in state space was motivated by the desire to take advantage of the inherent numerical robustness of state space methods over polynomial methods [8].

The first steps towards a simple state space solution to this problem was the introduction of the *Damped Riccati Algorithm* (DRA) in a recent paper [2]. This algorithm functions in the following way: Given the standard linear time invariant system

$$\begin{array}{rcll} \dot{x} & = & Ax & + \quad Bu \\ y & = & Cx, & \end{array} \tag{1}$$

with $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, and $C \in \mathbf{R}^{n,\ell}$, the DRA, under most circumstances, will compute a matrix $F \in \mathbf{R}^{m,n}$ such that the eigenvalues of the closed–loop matrix $A + BF$ are *damped, i.e.,* the eigenvalues of the closed–loop system have real parts greater in magnitude than their imaginary parts. Integral to this algorithm is the solution of the *Symmetric Damped Algebraic Riccati Equation (SDARE)*

$$X(A^2 - RS) + (A^{2T} - SR)X - X(AR + RA^T)X + (A^TS + SA) = 0, \tag{2}$$

and the *Skew-Symmetric Damped Algebraic Riccati Equation (SSDARE)*

$$Y(A^2 + RS) + (A^{2T} + SR)Y - Y(AR - RA^T)Y + (A^TS - SA) = 0, \tag{3}$$

as well as the more standard *Shifted Algebraic Riccati Equation* (SHARE)

$$N(A + \sigma I) + (A^T + \sigma I)N - NRN + S = 0. \tag{4}$$

Here, as for the rest of the paper $R := BB^T$ and $S := C^TC$. It was shown in [2] that if the SDARE and SSDARE are solvable, then the two feedbacks $F_X = -B^TX$ and $F_Y = -B^TY$ will produce closed–loop systems that are damped. Since the SDARE is a standard Riccati equation, standard solvability theory [3] can be applied to characterise when a solution exists. The close relationship between the solvability of the SSDARE and the SDARE described in [2] also provides the theoretical basis for the solvability of the SSDARE. Once a system is dampened, it may be stabilised by computing an additional feedback which symmetrically reflects the poles across the imaginary axis. This is accomplished by computing the stabilising solution to a degenerate SHARE with $S = 0$ and $\sigma = 0$. If $\sigma > 0$, then the closed–loop eigenvalues will not only be stable and damped, but have guaranteed a degree of stability σ, in effect blunting the cone in the left half–plane which contains the system poles.

In this paper, we introduce two parameters: one for the modification of the SDARE and SSDARE and one for the convex combination of the solutions of these two equations, and show that we again obtain dampening feedbacks.

2 Parameterised Damped Riccati Equations

In [2], it was shown that the solutions to the SDARE and the SSDARE are related to the stable invariant subspaces of the matrix product $H_X = H_2H_1$ and $H_Y = H_1H_2$, respectively, where

$$H_1 := \begin{bmatrix} -A & R \\ S & A^T \end{bmatrix}, \; H_2 := \begin{bmatrix} A & -R \\ S & A^T \end{bmatrix}. \tag{5}$$

If the columns of the $2n \times n$ matrix

$$T_s = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}, \tag{6}$$

span the invariant subspace corresponding to the stable eigenvalues of the $2n \times 2n$ Hamiltonian matrix H_X, then the symmetric solution X of the SDARE (2) (if it exists) is given by

$$X = T_{21}T_{11}^{-1}. \tag{7}$$

An analogous result holds for Y.

We will now introduce two parameters, one which scales the two equations and one that produces a convex combination of the solutions. Let

$$D := \begin{bmatrix} I & \\ & \bar{\alpha} I \end{bmatrix}$$

where $\alpha = e^{\imath\phi}, \phi \in [0, \pi]$, and consider the scaled matrices

$$\hat{H}_1(\alpha) = DH_1D^{-1},\ \hat{H}_2(\alpha) = DH_2D^{-1}.$$

as well as

$$H_Z(\alpha) = \hat{H}_2(\alpha)\hat{H}_1(\alpha),\ H_W(\alpha) = \hat{H}_1(\alpha)\hat{H}_2(\alpha). \tag{8}$$

Associated with these products are the *Hermitian Damped Algebraic Riccati Equation (HDARE)*

$$\begin{aligned} &Z(A^2 - \alpha^2 RS) + (A^{2T} - \bar{\alpha}^2 SR)Z \\ &\qquad -Z(\bar{\alpha}AR + \alpha RA^T)Z + (\bar{\alpha}A^TS + \alpha SA) = 0, \end{aligned} \tag{9}$$

and the *Skew-Hermitian Damped Algebraic Riccati Equation (SHDARE)*

$$\begin{aligned} &W(A^2 + \alpha^2 RS) + (A^{2T} + \bar{\alpha}^2 SR)W \\ &\qquad -W(\bar{\alpha}AR - \alpha RA^T)W + (\bar{\alpha}A^TS - \alpha SA) = 0, \end{aligned} \tag{10}$$

respectively.

The HDARE and the SHDARE have a very close relationship with the SDARE and the SSDARE; namely, the matrices H_X and H_Z are similar for all α and analogously the matrices H_Y and H_W are similar for all α. This close relationship is formalised in the following Corollary to Theorem 1 in [2].

Corollary 1 *Let $\alpha = e^{\imath\phi}, \phi \in [0, \pi]$. Suppose that there exists a Hermitian stabilising solution Z of (9) and a skew-Hermitian stabilising solution W of (10), then the closed-loop matrices $A - \bar{\alpha}RZ$ and $A - \alpha RW$ have eigenvalues within the closure of the damped region of the complex plane (excluding the point 0).*

Proof. The proof follows directly from that of Theorem 1 in [2] by noting the analogous roles of X and Z, and Y and W. $\square$

Remark 1 Despite the fact that a complex feedback is usually unrealisable for real systems, taking the real part of the feedback often suffices to dampen the system dynamics. Numerical experiments show that for the vast majority of randomly generated cases (999 out of 1000), $A - R\ \mathrm{Real}(\bar{\alpha}Z)$ and $A - R\ \mathrm{Real}(\alpha W)$ are damped. Indeed, for the two-by-two SISO case the dampedness of $A - R\ \mathrm{real}(\bar{\alpha}Z)$ and $A - R\ \mathrm{Real}(\alpha W)$ may be shown analytically.

As it happens, α is not the only parameter that may be varied without destroying the damping. Any convex combination of the two complementary solutions $\bar{\alpha}Z$ and αW produces a dampening controller as well. This is shown in the following theorem.

Theorem 2 *Let*

$$K = \beta\bar{\alpha}Z + (1-\beta)\alpha W. \tag{11}$$

with Z, W stabilising solutions of (9) and (10), respectively. For $\alpha = e^{\imath\phi}, \phi \in [0, \pi/4]$ and $\beta \in [0,1]$ the eigenvalues of $A - RK$ are within the closure of the damped region of the complex plane (excluding the point 0).

Proof. Consider the Hermitian matrices

$$M_Z = -(\alpha Z + \bar{\alpha}W)(A - \bar{\alpha}RW) = ZRZ - \bar{\alpha}A^T - \alpha ZA - S$$

and

$$M_W = -(\alpha Z + \bar{\alpha}W)(A - \alpha RZ) = WRW + \alpha A^T - \alpha A + S.$$

Lyapunov's Theorem [4] was used in Corollary 1 (by analogy from Theorem 1 of [2]) to show that the stability of $(A - \bar{\alpha}RZ)(A - \alpha RW)$ implies that M_Z and M_W are positive definite whenever S is positive definite. Introduce

$$\begin{aligned} M_K &:= \beta M_Z + (1-\beta)M_W \\ &= (\bar{\alpha}Z + \alpha W)(A - RK) \\ &= (A - RK)^H(\alpha Z + \bar{\alpha}W), \end{aligned} \tag{12}$$

where K is defined as in (11). Following the proof of Theorem 1 in [2], dampedness is proved if the following two conditions hold:

$$\begin{aligned} P_Z &= -(A - RK)^{2H}M_K - M_K(A - RK)^2 < 0 \\ M_K &> 0. \end{aligned} \tag{13}$$

Attending to the latter condition first, we note that since M_Z and M_W are positive definite then M_K will be positive definite for $\beta \in [0,1]$. With the substitution of the expressions in (12) for M_K in the first equation of (13) we have that

$$\begin{aligned} P_Z &= -(A-RK)^{2H}M_K - M_K(A-RK)^2 \\ &= -(A-RK)^H\Big((A-RK)^H(\alpha Z+\bar{\alpha}W)+ \\ &\qquad\qquad (\bar{\alpha}Z+\alpha W)(A-RK)\Big)(A-RK) \\ &= -(\bar{\alpha}^2+\alpha^2)(A-RK)^H\left(S+\beta ZRZ+(1-\beta)W^HRW\right)(A-RK) \\ &< 0 \end{aligned} \tag{14}$$

whenever S is positive definite. As in Theorem 1 of [2] it follows that the eigenvalues of $A-RK$ are in the interior of the damped region of the complex plane. If S is only positive semidefinite then by continuity it follows that the eigenvalues are in the closed damped region of the complex plane. □

3 Examples

In this section, we illustrate the effect of the parameters α and β on closed–loop eigenvalues.

In the first example, we validate the assertions in Remark 1. We start with a two by two matrix in controllable canonical form. Further, the system matrix is written in a form akin to that which describes a standard second order system in terms of its natural frequency ω and its damping factor ζ [7]:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}\omega^2 & -\omega\zeta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ R &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = 0. \end{aligned} \tag{15}$$

Note that the definition for the damping factor ζ is slightly modified; an undamped system as defined in this paper has $\zeta \in [0,1]$.

As this system is relatively simple, it is possible to get closed–form expressions for Z and W

$$\begin{aligned} Z &= \frac{\omega(\zeta^2-1)}{2-\zeta^2(\alpha+\bar{\alpha})^2}\begin{bmatrix} (\alpha+\bar{\alpha})\omega^2\zeta & 2\bar{\alpha}\omega \\ 2\alpha\omega & 2(\alpha+\bar{\alpha})\zeta \end{bmatrix}, \\ W &= \frac{\omega(\zeta^2-1)}{2+\zeta^2(\alpha-\bar{\alpha})^2}\begin{bmatrix} -(\alpha-\bar{\alpha})\omega^2\zeta & -2\alpha\omega \\ \bar{\alpha}\omega & -2(\alpha-\bar{\alpha})\zeta \end{bmatrix}. \end{aligned} \tag{16}$$

As mentioned earlier, the goal of this example is to illustrate the effect of the feedback on the projection of the *complex–valued* feedback matrix onto the field of *real numbers*. The paradigm of the parameterisation of a second order system in terms of its natural frequency and damping factor is again exploited; the closed–loop matrix $A - R\ \mathrm{real}(\bar{\alpha}Z)$ will be described in terms of ω_Z and ζ_Z, and likewise for the closed–loop matrix $A - R\ \mathrm{real}(\alpha W)$. These values are

$$\begin{aligned} \omega_Z^2 &= \frac{\omega^2\left(1-2\zeta^2\sin^2\phi\right)}{\left(1-2\zeta^2\cos^2\phi\right)} \quad , \quad \zeta_Z^2 = \frac{-2\zeta^2\cos^2 2\phi}{2-4\zeta^2+\zeta^4(2-\sin^2 2\phi)}, \\ \omega_W^2 &= \frac{\omega^2\left(1-2\zeta^2\cos^2\phi\right)}{\left(1-2\zeta^2\sin^2\phi\right)} \quad , \quad \zeta_W^2 = \frac{-2\zeta^2\cos^2 2\phi}{2-4\zeta^2+\zeta^4(2-\sin^2 2\phi)}. \end{aligned} \tag{17}$$

The formulae for the matrices Z and W indicate that the existence of the solutions depend on the location of the open–loop eigenvalues in the complex plane as well as the value of the parameter α; namely, the existence of the Z is assured if $2 - \zeta^2(\alpha + \bar{\alpha})^2 \neq 0$ and the existence of the W is assured if $2 + \zeta^2(\alpha - \bar{\alpha})^2 \neq 0$. The effect of varying $\alpha = e^{\imath\phi}$ on the existence of Z and W is evident in values of ω_Z and ω_W; these values tend to infinity as $1 - 2\zeta^2\cos^2\phi$ and $1 - 2\zeta^2\sin^2\phi$ go to zero, respectively.

It is also interesting to note that while the values of ζ_Z and ζ_W are equal and independent of ω, the values of ω_Z and ω_W are not equal nor are they independent of the value of ϕ and ζ. This might be counterintuitive, as one might expect the closed–loop eigenvalues to be reflected across the diagonal axes in the complex plane given that $S = 0$, thus preserving the value of ω under feedback. This does indeed occur when $\phi = \frac{\pi}{2}$, which corresponds the case when $H_Z = H_W$. In this case it should be noted that stable invariant subspaces of H_Z and H_W may be directly computed from $H_1 (= H_2)$ as $H_Z = H_W = H_1^2$.

Finally, to show that the closed-loop systems $A - R\ \mathrm{real}(\bar{\alpha}Z)$ and $A - R\ \mathrm{real}(\alpha W)$ are damped upon feedback, it is necessary to show that ζ_Z^2 and $\zeta_W^2 \notin (0,1]$ when $\zeta \in (0,1]$. First, note that $\zeta_Z^2 = \zeta_W^2 = 0$ when $\zeta = 0$ and that $\frac{\partial \zeta_Z^2}{\partial \zeta} < 0$ for $\zeta \in \frac{1}{2}\left(0, \min(\sec^2\phi, \csc^2\phi)\right)$ Secondly, note that $\zeta_Z^2 = \zeta_W^2 = 1$ when $\zeta = 1$ and that $\frac{\partial \zeta_Z^2}{\partial \zeta} < 0$ for $\zeta \in (\csc 2\phi, \frac{1}{2}\min(\sec^2\phi, \csc^2\phi))$. Since $\csc 2\phi \geq 1$, by continuity we have that $\zeta_Z^2, \zeta_W^2 \notin (0,1]$ when $\zeta \in (0,1]$, which is equivalent to asserting that $A - R\ \mathrm{real}(\bar{\alpha}Z)$ and $A - R\ \mathrm{real}(\alpha W)$ are damped.

In the next example, we show the effect on the closed–loop eigenvalues as parameter β is varied on $[0,1]$, with $\alpha = 1$. The closed-loop eigenvalues varying as a function of β are shown in Figure 2.

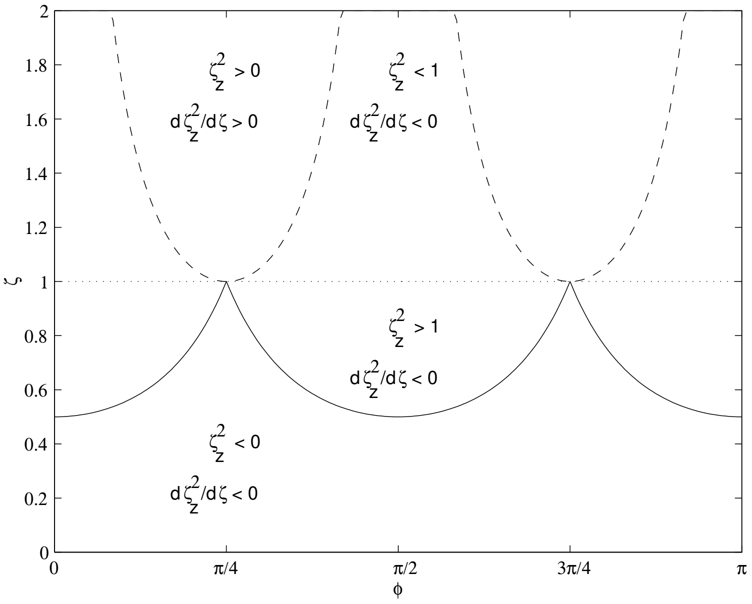

Figure 1: The dependence of ζ_Z^2 with the parameters $\alpha = e^{\imath\phi}$ and ζ

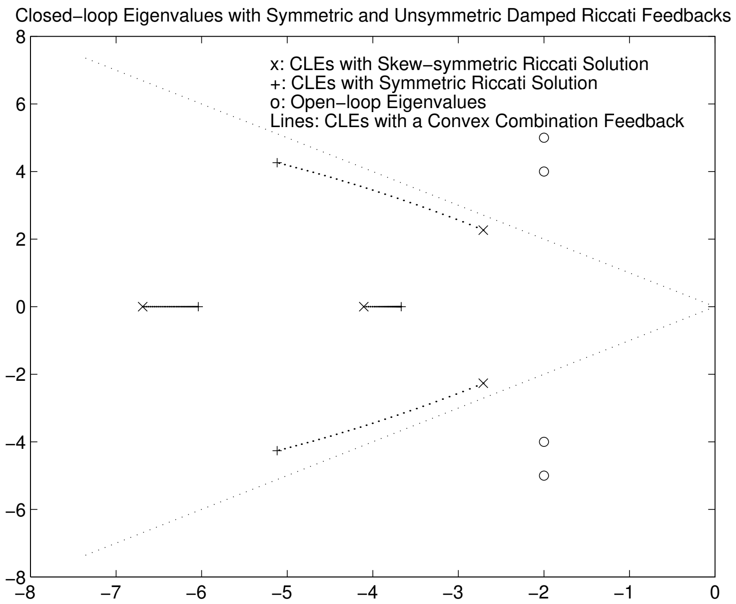

Figure 2: Varying the Closed-loop Eigenvalues with Parameter β

4 Concluding Remarks

In this paper we have extended to results of [2] on damped Riccati equations to include a family of dampening feedbacks that may be parameterised over two variables. This parameterisation may be of significance as it allows additional degrees of freedom with which other control criteria in addition to dampedness may be achieved.

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