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RESEARCH REPORT

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Probability distribution of an approximated function with uncertain argument

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Abstract

This contribution introduces a method for correct dealing with approximations in cases when argument of an approximated function is random variable. The approximative probability distribution is corrected to obtain a resulting probability distribution which incorporates a prior knowledge of approximation error.

Keywords: approximation, transformation of random variable

1 Introduction

In experimental sciences as well as in almost all the technical branches approximation of the real world is necessary. Typically, a function which expresses relationship among some quantities that are significant for the description of the reality is quite complicated and hard to evaluate. A simpler approximative function is therefore adopted instead. The difference between the original function and its approximation (approximation error) can usually be easily estimated and kept sufficiently small (for the given purpose).

A more sophisticated dealing with approximation is required when argument x of the original function t is measured with limited accuracy. The value of the function is therefore uncertain too. The uncertainty of the function value $t(x)$ expressed by means of probability density function $\bar{f}$ can be easily derived from the probability density function f of the argument x (see e.g. [4], Theorem 7.1.1, p. 172) :

$$\bar{f}(y) = f(t^{-1}(y)) \left| \frac{d}{dy} t^{-1}(y) \right| . \tag{1}$$

The formula (1) is not so easy to evaluate even when the function t is only slightly complicated. Therefore an approximative function is used instead of the original function t as in the previous case. But the difference between the exact probability distribution and the approximative probability distribution is neglected without any assessment of its magnitude. So the decisions made on the approximative probability distribution cannot be reliable.

This difficulty is resolved in this contribution. The approximative probability distribution is corrected with respect to the approximation error and the outcoming probability distribution (corrected distribution) is studied. The core of this idea was published in a concise form in [3].

2 Motivation examples

Functions are being approximated mainly for two reasons. Firstly, the function values can be gained only by measurement of the corresponding physical quantity, e.g. depth of a sea under an arbitrary point on the sea level, temperature of some body in chosen points, resistance along a non-homogenous wire etc. The second type of functions can be expressed in form of a mathematical formula, but the evaluation of the function is very complicated and time consuming. There are also many practical examples, especially from mathematical physics: solution of nonlinear algebraic equations, partial differential equations, integral equations, etc.

Let us have a short look at the most illustrative examples of both types.

2.1 Depth determination problem

An illustrative example of the first type is problem of a captain of a ship who wants to know the depth under the ship and the accuracy of the depth.

He cannot measure the depth directly, but he has got a digital elevation model of the sea bottom (or nautical map with depth dimensions) which gives him the depth for any point on the sea level (using the map some interpolation has to be made). The position of the ship was measured with certain accuracy, so the depth and its accuracy can be derived with the aid of the digital elevation model of the sea bottom. Even the probability density function $\bar{f}$ of the depth can be easily computed by means of two-dimensional version of the formula (1) if probability density function f of the ship location is given, because the function which forms the digital elevation model t is usually simple interpolation polynomial. But the digital elevation model is only approximation of the true shape of the bottom. The approximation error should be put to use in the computation together with the uncertainty of the ship location. The method proposed below exploits the information on the approximation error so that the correct probability distribution of the depth outcomes from the procedure.

2.2 Geometric transformation of a remotely sensed image

As an example of the second type let us consider transformation of coordinates of points in the satellite image to some cartographic coordinate system. The transformation is a projective transformation modified by some geometric distorsions (earth curvature and rotation, optical and mechanical nonlinearities of the scanner construction, etc.) Under some assumptions the transformation is close to affinity, i.e. the transformation is linear. When a probability density function f of a point in the image is given, the corresponding probability density function $\bar{f}$ of its cartographic coordinates can be determined by means of a formula similar to (1). The estimation of the difference between the exact transformation and affinity should be considered.

3 Formulation of the problem

Let us consider a function

$$b : \mathcal{X} \to \mathbb{R} : x \mapsto b(x) \ ,$$

which is approximated by another function

$$a : \mathcal{X} \to \mathbb{R} : x \mapsto a(x) \ ,$$

where

- $\diamond$ $\mathcal{X} \subset \mathbb{R}$,
- $\diamond$ function a is strictly monotonous on $\mathcal{X}$,
- $\diamond$ $a(x) \doteq b(x) \Leftarrow x \in \mathcal{X}$, i.e.

$$\exists K \in \mathbb{R} \, , K > 0 : \forall \, x \in \mathcal{X} : |b(x) - a(x)| < K \ .$$

Function a is given analytically by means of an explicit formula. The monotonicity of function a is assumed only for simplicity of subsequent expressions. This assumption is not essential in general.

Function b is known only in some discrete points $x_0 \, , \, x_1 \, , \ldots \, , \, x_n \in \mathcal{X}$. The quantity x is assigned by a random variable X with probability density function (pdf)

$$f : \mathcal{X} \to \mathbb{R} : x \mapsto f(x) \ .$$

The support of pdf f is a constrained set U_f:

$$U_f \stackrel{\text{def}}{=} \{ x \in \mathcal{X} \, | \, f(x) > 0 \} \ . \tag{2}$$

The support U_f has to be covered by the smallest interval containing all the points $x_0 \, , \, x_1 \, , \ldots \, , \, x_n$. Set U_f is typically formed by a real interval.

The approximation error

$$r(x) \stackrel{\text{def}}{=} b(x) - a(x) \tag{3}$$

is also treated as a random variable probability distribution of which is to be proposed with the aid of points $[x_i \, , \, r(x_i)] \, , \, i \in \{0 \, , \, 1 \, , \ldots \, , \, n\}$. Let us denote such a random variable C_x. Its probability distribution is conditional distribution whith density function

$$p(\, . \, | \, x) : r(\mathcal{X}) \to \mathbb{R} : c \mapsto p(c \, | \, x) \ .$$

Probability distribution of the superposition $a(X) + C_x$ has to be found.

4 Problem solution

4.1 General idea

Let us denote the pdf's of the random variables Y, $a(X)$ and the joint probability density function of the pair $[C_x\,,\, a(X)]$:

- $Y \ldots \bar{f} : \mathcal{X} \to \mathbb{R} : y \mapsto \bar{f}(y)\,,$
- $a(X) \ldots f_a : a(\mathcal{X}) \to \mathbb{R} : z \to f_a(z) = \dfrac{f(a^{-1}(z))}{\left|\dfrac{da}{dx}(a^{-1}(z))\right|}\,,$
- $[C_x\,, a(X)] \ldots h : \mathbb{R} \times a(\mathcal{X}) \to \mathbb{R} : [c\,, z] \mapsto h(c\,, z) = p(c\,|\,a^{-1}(z))\, f_a(z).$

Then the pdf $\bar{f}$ is given by the known formula for the probability distribution of the sum of random variables (see e.g. [4], Theorem 7.3.1, p. 180):

$$\begin{aligned} \bar{f}(y) &= \int_{a(\mathcal{X})} h(y - z\,, z)\, dz = \\ &= \int_{a(\mathcal{X})} p(y - z\,|\,a^{-1}(z))\, f_a(z)\, dz = \\ &= \int_{\mathcal{X}} p(y - a(x)\,|\,x)\, f(x)\, dx\ . \end{aligned} \tag{4}$$

4.2 A prior probability distribution of the approximation error

Integral (4) can be exactly evaluated only if a suitable prior probability density function $p(\,.\,|\,x)$ is set up. It seems natural to define it as uniform on certain constrained interval. Later in this section we will see that this option is not sufficient for a more general cases. $p(\,.\,|\,x)$ will be composed from several uniform distributions.

4.2.1 Uniform prior distribution

Let us denote the bounding points of the domain of $p(\,.\,|\,x)$ by means of a couple of functions (border functions):

$$s\,, S : \mathcal{X} \to \mathbb{R}\,; \quad \forall\, x \in \mathcal{X} :\ s(x) < S(x)\ .$$

Then

$$p(c\,|\,x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{S(x)-s(x)} & \Leftarrow c \in (s(x)\,,\,S(x))\,, \\ 0 & \Leftarrow c \notin (s(x)\,,\,S(x))\,. \end{cases} \tag{5}$$

This definition implies a new range of integration in (4):

$$\omega_a(y) \stackrel{\text{def}}{=} \{x \in \mathcal{X}\,|\,y - a(x) \in (s(x)\,,\,S(x))\}\ . \tag{6}$$

Using these options in integral (4), the pdf $\bar{f}$ can be expressed as

$$\bar{f}(y) = \int_{\omega_a(y)} \frac{1}{S(x)-s(x)}\, f(x)\,dx \tag{7}$$

Remark: The range of integration is better to express as $\omega_a(y) \cap U_f$, because this form allows to evaluate the integral analytically. Equation (7) with only $\omega_a(y)$ is mathematically correct as well, since the integrand is zero outside the set $\omega_a(y) \cap U_f$. This remark is valid for the all ranges of integration in the sequel.

Two different cases of setting the interval $(s(x)\,,\,S(x))$ will be disscussed now. In the both cases the exact function b is known only at two points $x_0\,,\,x_1 \in \mathcal{X}$. Let us denote

$$\begin{aligned} x^- &\stackrel{\text{def}}{=} \min\{x_0\,,\,x_1\}\,, \\ x^+ &\stackrel{\text{def}}{=} \max\{x_0\,,\,x_1\}\,. \end{aligned} \tag{8}$$

The prior information about function r in $(x^-\,,\,x^+)$ is given by inequality:

$$\min\{r(x_0)\,,\,r(x_1)\} < r(x) < \max\{r(x_0)\,,\,r(x_1)\}\ . \tag{9}$$

Unconditional uniform prior distribution When no other information than (9) is available, the noninformative prior distribution should be admitted. It means that the border functions $s\,,S$ are constant in $(x^-\,,\,x^+)$. In such a case the prior pdf $p(\,.\,|\,x)$ is not conditional. Graph of function r then lies somewhere in a rectangle determined by its opposite corners $[x_0\,,\,r(x_0)]$, $[x_1\,,\,r(x_1)]$ (see Fig. 1). Values $x_0\,,\,x_1$ represent bounding points of an interval in $\mathcal{X}$ which covers the support of pdf f.

Thus for all $x \in (x^- \, , \, x^+)$ the next equalities hold:

$$
\begin{aligned}
s(x) &= r(x^{\mp})\,, \\
S(x) &= r(x^{\pm})\,, \qquad\qquad (10) \\
\pm\,,\, \mp \in \{+\,,\,-\}\,, \pm &\neq \mp\,,
\end{aligned}
$$

$$
\pm = \begin{cases} + & \Leftarrow r \text{ is increasing on } \mathcal{X}\,, \\ - & \Leftarrow r \text{ is decreasing on } \mathcal{X}\,. \end{cases} \qquad (11)
$$

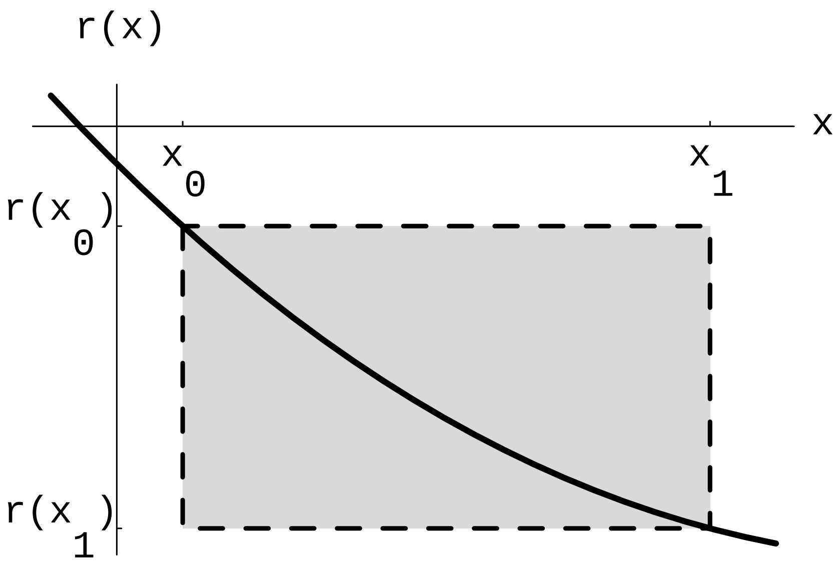

Figure 1: True approximation error r (solid line) and area where the graph of function r is expected to lie in (gray rectangle)

After substitution of (10) in (7) and (6) we receive:

$$
\bar{f}(y) = \frac{1}{r(x^{\pm}) - r(x^{\mp})} \int_{\omega_a(y)} f(x)\,dx \;, \qquad (12)
$$

$$
\begin{aligned}
\omega_a(y) &= \Big\{ x \in (x^- \,,\, x^+) \,|\, y - a(x) \in (r(x^{\mp})\,,\, r(x^{\pm})) \Big\} = \\
&= (a^{-1}(y - r(x^{*\pm})\,,\, a^{-1}(y - r(x^{*\mp}))\;,\; * \in \{+\,,\,-\}\;, \qquad (13) \\
& \quad * = \begin{cases} + & \Leftarrow a \text{ is increasing on } \mathcal{X}\,, \\ - & \Leftarrow a \text{ is decreasing on } \mathcal{X}\,. \end{cases}
\end{aligned}
$$

The expression in superscript of x should be evaluated according to the following rules:

$$
\begin{aligned}
+- = -+ = -\;, \\
++ = -- = +\;.
\end{aligned}
$$

The probability density function $\bar{f}$ can finally be expressed:

$$
\bar{f}(y) = \frac{1}{|r(x_1) - r(x_0)|} \int_{\omega_a(y)} f(x)\,dx\;. \tag{14}
$$

Conditional uniform prior distribution on 2 points The knowledge of the approximation error is usually more informative than in the previous case. If function r is monotonous on (x^-, x^+) and the range of derivative of r is constrained, the area of expected graphs of function r can be smaller than the rectangle in Fig. 1. Let us assume that the border functions $s\,,S$ are simple polynomials:

$$
\begin{aligned}
s(x) = r(x^{\mp}) \pm k(\pm(x - x^{\mp}))^m\;, \\
S(x) = r(x^{\pm}) \mp k(\mp(x - x^{\pm}))^m\;, \\
m \in \mathbb{N}, \quad m \geq 2\,, \quad k = \frac{r(x^+) - r(x^-)}{(x^+ - x^-)^m}\;.
\end{aligned} \tag{15}
$$

The symbols $\pm\,,\mp$ have the same meaning as in the previous paragraph (formula (11)).

The polynomials s, S intersect each other in points $[x_0\,,\, r(x_0)]$, $[x_1\,,\, r(x_1)]$ (see Fig. 2).

The order of polynomials m controls the magnitude of the area between graphs of functions $s\,,S$. The higher m, the larger area. This area becomes rectangular as in the previous case when $m \to \infty$.

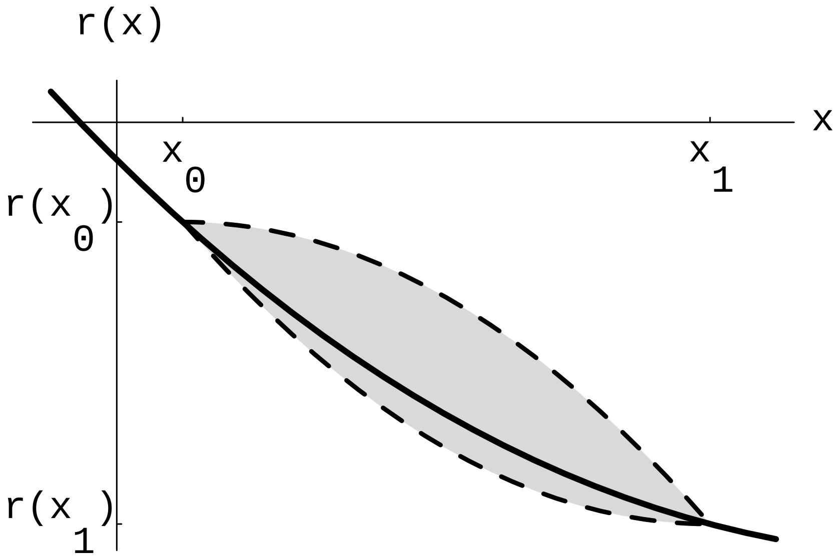

Figure 2: Prior knowledge of approximation error r (gray area) for conditional uniform prior distribution

The length of interval $(s(x), S(x))$ is:

$$S(x) - s(x) = r(x^{\pm}) - r(x^{\mp}) \mp k(\mp(x - x^{\pm}))^m \mp k(\pm(x - x^{\mp}))^m =$$

$$\begin{aligned}
&= r(x^{\pm}) - r(x^{\mp}) \mp k((\mp(x - x^{\pm}))^m + (\pm(x - x^{\mp}))^m = \\
&= r(x^{\pm}) - r(x^{\mp}) \mp \frac{r(x^+) - r(x^-)}{(x^+ - x^-)^m}((x^+ - x)^m + (x - x^-)^m) = \\
&= \pm(r(x^+) - r(x^-)) \mp \frac{r(x^+) - r(x^-)}{(x^+ - x^-)^m}((x^+ - x)^m + (x - x^-)^m) = \\
&= \mp \frac{r(x^+) - r(x^-)}{(x^+ - x^-)^m}((x^+ - x)^m + (x - x^-)^m - (x^+ - x^-)^m) = \\
&= -|k|\,((x^+ - x)^m + (x - x^-)^m - (x^+ - x^-)^m)\,.
\end{aligned}$$

For $m = 2$ we can use the equality

$$(x^+ - x^-)^2 = (x^+ - x)^2 + 2(x^+ - x)(x - x^-) + (x - x^-)^2$$

and simplify the length:

$$\begin{aligned} S(x) - s(x) &= -|k|\left((x^+ - x)^2 + (x - x^-)^2 - (x^+ - x^-)^2\right) = \\ &= 2|k|\,(x^+ - x)(x - x^-)\ . \end{aligned} \tag{16}$$

Hence a prior pdf $p(\,.\,|\,x)$ is:

$$p(c\,|\,x) = \begin{cases} \dfrac{1}{2|k|\,(x^+ - x)(x - x^-)} & \Leftarrow c \in (s(x)\,,\, S(x))\,, \\ 0 & \Leftarrow c \notin (s(x)\,,\, S(x))\,, \end{cases} \tag{17}$$

where $s(x)\,,\, S(x)$ are given by (15) with $m = 2$. Functions $s(x)\,,\, S(x)$ defined in (15) changes the set $\omega_a(y)$. After substitutions (15) with $m = 2$ in (6) the new set $\omega_a(y)$ follows:

$$\begin{aligned} \omega_a(y) = \{x \in \mathcal{X}\,|\, \\ y - a(x) \in (r(x^\mp) \pm k(\pm(x - x^\mp))^2\,,\, r(x^\pm) \mp k(\mp(x - x^\pm))^2))\}\ . \end{aligned} \tag{18}$$

The resulting pdf $\bar{f}$ then has form

$$\bar{f}(y) = \int_{\omega_a(y)} \frac{1}{2|k|\,(x^+ - x)(x - x^-)}\, f(x)\, dx\ . \tag{19}$$

After factorization of the integral in (19) to partial fractions we finally have:

$$\bar{f}(y) = \frac{x^+ - x^-}{2\,|r(x^+) - r(x^-)|}\left(\int_{\omega_a(y)} \frac{f(x)}{x - x^-}\, dx - \int_{\omega_a(y)} \frac{f(x)}{x - x^+}\, dx\right)\ . \tag{20}$$

4.2.2 Conditional uniform prior distribution on 3 points

The previous procedure concluded by formula (20) cannot, unfortunately, be generalized to an arbitrary number of points $x_i\,,\, i \in \{0\,,\,1\,,\ldots,\,n\}$. The polygon with vertices $[x_i\,,\, r(x_i)]\,,\, i \in \{0\,,\,1\,,\ldots,\,n\}$ has not continuous first derivatives which causes discontinuity of the pdf $\bar{f}$. Therefore the polygon should be substituted by a smooth curve. A polynomial interpolation could be used, but polynomials of higher orders may cause computational problems. Let us start with 3 points $[x_0\,, r(x_0)]\,, [x_1\,, r(x_1)]\,, [x_2\,, r(x_2)]\,, x_0 < x_1 < x_2$ and try interpolation of the second order. We will use the Newton's interpolating polynomials (see e.g. [1], paragraph 25.2.26) of the first and second orders :

$$
\begin{aligned}
q_1(x) &\stackrel{\text{def}}{=} r(x_0) + A_{01}(x - x_0)\ , \quad A_{01} \stackrel{\text{def}}{=} \frac{r(x_1) - r(x_0)}{x_1 - x_0}\ , \\
q_2(x) &\stackrel{\text{def}}{=} r(x_0) + A_{01}(x - x_0) + B_1(x - x_0)(x - x_1)\ , \\
& \quad B_1 \stackrel{\text{def}}{=} \frac{A_{01} - A_{12}}{x_0 - x_2}\ , \\
& \quad \text{for } \forall\, i\,,\, j \in \{0\,,\, 1\,,\, 2\} :\ A_{ij} \stackrel{\text{def}}{=} \frac{r(x_j) - r(x_i)}{x_j - x_i}\ .
\end{aligned}
\tag{21}
$$

Starting from the previous case we will construct one couple of the border functions similarly to $s\,, S$ for $x^- = x_0\,,\ x^+ = x_1$ and another couple for $x^- = x_1\,,\ x^+ = x_2$. Since the linear coefficient A_{01} occures in front of the integral in (20) ($A_{01} = (r(x^+) - r(x^-))/(x^+ - x^-)$), we can consider the border functions $s\,, S$ as referring to the diagonal line $q_1(x)$.

Firstly, we substitute the line $q_1(x)$ by the parabola $q_2(x)$ and simultaneously keep the references to the border functions unchanged in the interval $(x_0\,,\, x_1)$. In this way a new couple of border functions $s_0\,, S_0$ can be introduced for $\forall\, x \in (x_0\,,\, x_1)$:

$$
\begin{aligned}
s_0(x) - q_2(x) &= s(x) - q_1(x)\ , \\
S_0(x) - q_2(x) &= S(x) - q_1(x)\ .
\end{aligned}
$$

Hence for $\forall\, x \in (x_0\,,\, x_1)$:

$$
\begin{aligned}
s_0(x) &= s(x) + q_2(x) - q_1(x) = s(x) + B_1(x - x_0)(x - x_1)\ , \\
S_0(x) &= S(x) + q_2(x) - q_1(x) = S(x) + B_1(x - x_0)(x - x_1)\ .
\end{aligned}
\tag{22}
$$

Secondly, we consider the other diagonal line

$$
\tilde{q}_1(x) \stackrel{\text{def}}{=} r(x_2) + A_{21}(x - x_2)
$$

for $\forall\, x \in (x_1\,,\, x_2)$. The parabola $q_2(x)$ can be expressed in equivalent form:

$$
q_2(x) = r(x_2) + A_{21}(x - x_2) + B_1(x - x_2)(x - x_1)\ .
$$

For $x^- = x_1\,,\; x^+ = x_2$ we introduce auxillary functions $\tilde{s}\,,\, \tilde{S}$ that are defined as $s\,,\, S$ in (15). Then, analogically to (22), the new couple of border functions $s_1\,,\, S_1$ can be defined for $\forall\, x \in (x_1\,,\, x_2)$:

$$
\begin{aligned}
s_1(x) &= \tilde{s}(x) + q_2(x) - \tilde{q}_1(x) = \tilde{s}(x) + B_1(x - x_2)(x - x_1)\,, \\
S_1(x) &= \tilde{S}(x) + q_2(x) - \tilde{q}_1(x) = \tilde{S}(x) + B_1(x - x_2)(x - x_1)\,. \qquad (23)
\end{aligned}
$$

Now, a prior pdf $p(\,.\,|\,x)$ can be expressed:

$$
p(\,.\,|\,x) = \begin{cases} p_0(\,.\,|\,x) & \Leftarrow x \in (x_0\,,\, x_1)\,, \\ p_1(\,.\,|\,x) & \Leftarrow x \in (x_1\,,\, x_2)\,, \end{cases} \qquad (24)
$$

where for $\forall\, i \in \{0\,,\, 1\}$:

$$
p_i(c\,|\,x) = \begin{cases} \dfrac{1}{S_i(x) - s_i(x)} & \Leftarrow c \in (s_i(x)\,,\, S_i(x))\,, \\ 0 & \Leftarrow c \notin (s_i(x)\,,\, S_i(x))\,. \end{cases} \qquad (25)
$$

Finally, the corrected pdf $\bar{f}(y)$ can be derived.

$$
\begin{aligned}
\bar{f}(y) &= \int_{\mathcal{X}} p(y - a(x)\,|\,x)\, f(x)\, dx = \\
&= \int_{x_0}^{x_1} p_0(y - a(x)\,|\,x)\, f(x)\, dx + \\
&+ \int_{x_1}^{x_2} p_1(y - a(x)\,|\,x)\, f(x)\, dx = \\
&= \frac{x_1 - x_0}{2\,|r(x_1) - r(x_0)|} \left(\int_{\omega_{a,0}(y)} \frac{f(x)}{x - x_0}\, dx - \int_{\omega_{a,0}(y)} \frac{f(x)}{x - x_1}\, dx \right) + \\
&+ \frac{x_2 - x_1}{2\,|r(x_2) - r(x_1)|} \left(\int_{\omega_{a,1}(y)} \frac{f(x)}{x - x_1}\, dx - \int_{\omega_{a,1}(y)} \frac{f(x)}{x - x_2}\, dx \right)\,. \qquad (26)
\end{aligned}
$$

The ranges of integration $\omega_{a,i}(y)$, $i \in \{0\,,\, 1\}$ are given analogically to (6) by the formula:

$$
\omega_{a,i}(y) = \{x \in (x_i\,,\, x_{i+1})\,|\, y - a(x) \in (s_i(x)\,,\, S_i(x))\}\;. \qquad (27)
$$

4.2.3 Conditional prior distribution on a number of points

When the exact function b is known in $n+1$ points x_i , $i \in \{0\,, 1\,, \ldots\,, \; n-1\}$, a prior pdf $p(\,.\,|\,x)$ could be expressed analogically to (24) with the aid of partial pdf's $p_i(\,.\,|\,x)$:

$$\text{for } \forall\, i \in \{0\,, 1\,, \ldots\,, \; n-1\} : p(\,.\,|\,x) = p_i(\,.\,|\,x) \Leftarrow x \in (x_i\,, \, x_{i+1})\,. \qquad (28)$$

But for each interval $(x_i\,, \, x_{i+1})\,,\; i \in \{1\,, 2\,, \ldots\,, \; n-2\}$ (all the intervals except the first one and the last one) two different interpolating parabolas $q_2(x)$ exist. Therefore two different prior pdf's can be constructed by the method from the previous paragraph. It means that in the Fig. 3 two different areas (gray chains) can be drawn.

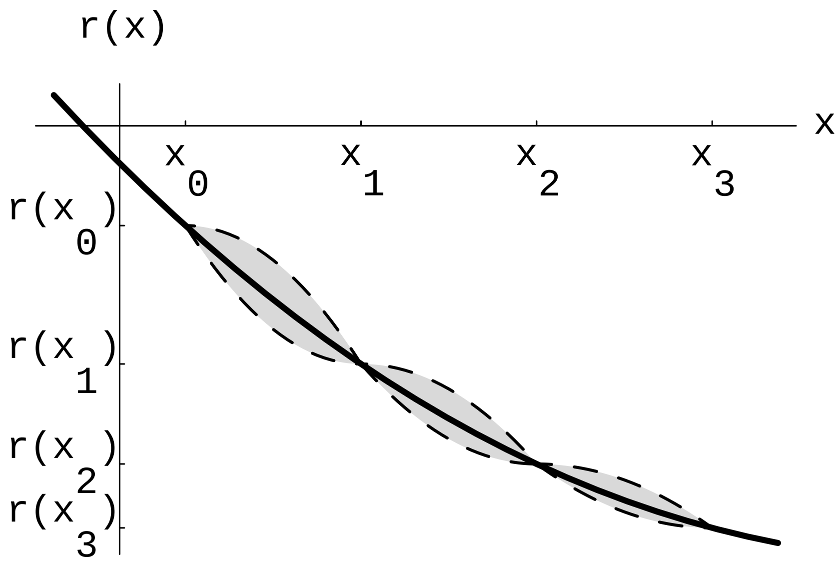

Figure 3: General case of a prior knowledge of approximation error

One of the parabolas intersects the graph of function r in points x_{i-1}, x_i, x_{i+1} and the other one in points x_i, x_{i+1}, x_{i+2}. The coresponding border functions s, S will be denoted as s_i^-, S_i^-, resp. s_i^+, S_i^+ and, similarly, prior pdf's $p_i^-(\,.\,|\,x)$, resp. $p_i^+(\,.\,|\,x)$. The new border functions can be determined by the same way as in (22), (23) if we admit the next notation:

for $\forall\, i \in \{0\,,\, 1\,, \ldots,\, n-1\}$:

$$x_i^- \stackrel{\text{def}}{=} x_i\,, \quad x_i^+ \stackrel{\text{def}}{=} x_{i+1}\,, \quad k_i \stackrel{\text{def}}{=} \frac{r(x_{i+1}) - r(x_i)}{(x_{i+1} - x_i)^2}\,,$$

for $\forall\, i \in \{1\,,\, 2\,, \ldots,\, n-2\}\,,\ \square \in \{-\,,\,+\}$:

$$\begin{aligned} s_i^\square(x) &= r(x_i^\mp) \pm k_i(\pm(x - x_i^\mp))^2 + B_i^\square(x - x_i)(x - x_{i+1})\,, \\ S_i^\square(x) &= r(x_i^\pm) \mp k(\mp(x - x_i^\pm))^2 + B_i^\square(x - x_i)(x - x_{i+1})\,, \end{aligned} \tag{29}$$

where

$$B_i^\square = \frac{\dfrac{r(x_i^-) - r(x_i^+)}{x_i^- - x_i^+} - \dfrac{r(x_{i\square 1}^-) - r(x_{i\square 1}^+)}{x_{i\square 1}^- - x_{i\square 1}^+}}{x_i^{-\square} - x_{i\square 1}^\square}$$

and the symbols $\pm$, $\mp$ have the same meaning as it was introduced in (11).

Then, similarly, the prior pdf's can be determined as in (25):

$$p_i^\square(c\,|\,x) = \begin{cases} \dfrac{1}{S_i^\square(x) - s_i^\square(x)} & \Leftarrow c \in (s_i^\square(x)\,,\, S_i^\square(x))\,, \\ 0 & \Leftarrow c \notin (s_i^\square(x)\,,\, S_i^\square(x))\,. \end{cases}$$

The length $S_i^\square(x) - s_i^\square(x)$ can be derived analogically to $S(x) - s(x)$ in (16).

$$S_i^\square(x) - s_i^\square(x) = 2\,|k_i|\,(x_{i+1} - x)(x - x_i) \quad \text{for } \forall\, \square \in \{-\,,\,+\}\,. \tag{30}$$

Hence

$$p_i^\square(c\,|\,x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{2\,|k_i|\,(x_{i+1} - x)(x - x_i)} & \Leftarrow c \in (s_i^\square(x)\,,\, S_i^\square(x))\,, \\ 0 & \Leftarrow c \notin (s_i^\square(x)\,,\, S_i^\square(x))\,. \end{cases} \tag{31}$$

The most natural way how to combine these two pdf's is their mixture with weights $w_i^-(x)$, $w_i^+(x)$:

$$p(\,.\,|\,x) = w_i^-(x)\; p_i^-(\,.\,|\,x) + w_i^+(x)\; p_i^+(\,.\,|\,x)\,, \tag{32}$$

$$\text{for } \forall\, x \in (x_i\,,\, x_{i+1}) : w_i^-(x) + w_i^+(x) = 1\ .$$

The nature of such a mixture can be seen by inferring it from the theorem of total probability. It is shown in [2].

The weights $w_i^{\square}(x)$, $\square \in \{-\,,+\}$ should be properly designed. Weight $w_i^{\square}(x)$ represents a probability, that appearance of a point $[x\,,\, r(x)]$ between points $[x_i\,,\, r(x_i)]$, $[x_{i+1}\,,\, r(x_{i+1})]$ is controlled by the probability distribution with pdf $p_i^{\square}(\,.\,|\,x)$. Therefore the following condition must be held:

$$\text{for } \forall\, x',\, x'' \in (x_i\,,\, x_{i+1}) : x' < x'' \Rightarrow w_i^-(x') > w_i^-(x'')\ ,\ w_i^+(x') < w_i^+(x'')\ .$$

We set up the weights to be linear in x because it is convenient from the computational point of view.

$$w_i^-(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}\,, \quad w_i^+(x) = \frac{x - x_i}{x_{i+1} - x_i}\ .$$

Using these weights and the prior pdf's from (31) in (32) we get a prior pdf $p_i^+(\,.\,|\,x)$ in form:

$$\begin{aligned} p(c\,|\,x) &= \frac{x_{i+1} - x}{x_{i+1} - x_i} \cdot \frac{1}{2\,|k_i|\,(x_{i+1} - x)(x - x_i)} + \\ &+ \frac{x - x_i}{x_{i+1} - x_i} \cdot \frac{1}{2\,|k_i|\,(x_{i+1} - x)(x - x_i)} = \end{aligned}$$

$$= \frac{x_{i+1} - x_i}{2\,|r(x_{i+1}) - r(x_i)|\,(x - x_i)} + \frac{x_{i+1} - x_i}{2\,|r(x_{i+1}) - r(x_i)|\,(x_{i+1} - x)} \tag{33}$$

This result is, however, valid only for

$$c \in (s_i^-(x)\,,\, S_i^-(x)) \cap (s_i^+(x)\,,\, S_i^+(x))\ .$$

Outside the intersection

$$\gamma^{\cap}(x) \stackrel{\text{def}}{=} (s_i^-(x)\,,\, S_i^-(x)) \cap (s_i^+(x)\,,\, S_i^+(x))$$

one of the pdf's $p_i^-(\,.\,|\,x)\,,\, p_i^+(\,.\,|\,x)$ is zero and one of the terms in (33) vanishes. With the auxillary notation

$$\begin{aligned} \gamma^{\ominus}(x) &\stackrel{\text{def}}{=} (s_i^-(x)\,,\, S_i^-(x)) \setminus \gamma^{\cap}(x)\ , \\ \gamma^{\oplus}(x) &\stackrel{\text{def}}{=} (s_i^+(x)\,,\, S_i^+(x)) \setminus \gamma^{\cap}(x)\ . \end{aligned}$$

and using the linear coefficient $A_{i,i+1}$ defined as in (21) but for $\forall\, i \in \{0\,,\,1\,,\ldots,\,n-1\}$, the complete form of the pdf $p_i(\,.\,|\,x)$ can be expressed:

$$p_i(c\,|\,x) = \begin{cases} \dfrac{1}{2\,|A_{i,i+1}|\,(x-x_i)} & \Leftarrow c \in \gamma^{\ominus}(x)\,, \\ \dfrac{1}{2\,|A_{i,i+1}|\,(x-x_i)} + \dfrac{1}{2\,|A_{i,i+1}|\,(x_{i+1}-x)} & \Leftarrow c \in \gamma^{\cap}(x)\,, \\ \dfrac{1}{2\,|A_{i,i+1}|\,(x_{i+1}-x)} & \Leftarrow c \in \gamma^{\oplus}(x)\,. \end{cases} \tag{34}$$

The prior pdf $p_i(\,.\,|\,x)$ can have a better arranged form if special functions

$$\varphi_i\,,\,\psi_i : r(\mathfrak{X}) \to \mathbb{R} \quad \text{for} \quad \forall\, i \in \{1\,,\,2\,,\ldots,\,n-2\}$$

are introduced.

$$p_i(c\,|\,x) = \varphi_i(c)\,\frac{1}{x-x_i} + \psi_i(c)\,\frac{1}{x-x_{i+1}} \tag{35}$$

The functions $\varphi_i\,,\,\psi_i$ are constant in intervals $\gamma^{\ominus}(x)$, $\gamma^{\cap}(x)$, $\gamma^{\oplus}(x)$. These constants can be evaluated after comparison of (34) and (35). Let us denote them for $\forall \odot \in \{\ominus\,,\,\cap\,,\,\oplus\}$:

$$\begin{aligned} \varphi_i^{\odot} &\overset{\text{def}}{=} \varphi_i(c) \quad \Leftarrow c \in \gamma^{\odot}(x)\;, \\ \psi_i^{\odot} &\overset{\text{def}}{=} \psi_i(c) \quad \Leftarrow c \in \gamma^{\odot}(x)\;. \end{aligned}$$

Then

$$\begin{aligned} \varphi_i^{\oplus} &= \psi_i^{\ominus} = 0\;, \\ \varphi_i^{\cap} &= -\psi_i^{\cap} = \varphi_i^{\ominus} = -\psi_i^{\oplus} = \frac{1}{2\,|A_{i,i+1}|}\;. \end{aligned} \tag{36}$$

With the aid of (35) we introduce an auxillary function $\tilde{f}$:

$$\tilde{f}(y) = \sum_{i=1}^{n-2}\left(\int_{x_i}^{x_{i+1}} \varphi_i(y-a(x))\,\frac{f(x)}{x-x_i}\,dx + \int_{x_i}^{x_{i+1}} \psi_i(y-a(x))\,\frac{f(x)}{x-x_{i+1}}\,dx\right)$$

to write the corrected pdf in concise form:

$$
\begin{aligned}
\bar{f}_n(y) &= \frac{1}{2\,|A_{01}|}\left(\int_{\omega_{a,0}(y)} \frac{f(x)}{x-x_0}\,dx - \int_{\omega_{a,0}(y)} \frac{f(x)}{x-x_1}\,dx\right) + \\
&+ \tilde{f}(y) + \\
&+ \frac{1}{2\,|A_{n-1,n}|}\left(\int_{\omega_{a,n-1}(y)} \frac{f(x)}{x-x_1}\,dx - \int_{\omega_{a,n-1}(y)} \frac{f(x)}{x-x_2}\,dx\right) \qquad (37)
\end{aligned}
$$

where $\omega_{a,i}(y)$ for $i \in \{0\,,\, n-1\}$ are given by (27) with substitutions

$$
s_0 = s_0^+\,,\ S_0 = S_0^+\,, \quad s_{n-1} = s_{n-1}^-\,,\ S_{n-1} = S_{n-1}^- \qquad (38)
$$

and (29). Using auxillary sets

$$
\begin{gathered}
\omega_{a,i}^{\odot} = \left\{x \in (x_i\,,\, x_{i+1})\,|\, y - a(x) \in \gamma_i^{\odot}(x)\right\} \\
\text{for } \forall\, i \in \{0\,,\, 1\,, \ldots,\, n-1\}\,,\ \forall \odot \in \{\ominus\,,\, \cap\,,\, \oplus\}
\end{gathered}
$$

the auxillary function $\tilde{f}$ can be further simplified :

$$
\begin{aligned}
\tilde{f}(y) &= \sum_{i=1}^{n-2}\left(\int_{x_i}^{x_{i+1}} \varphi_i(y-a(x))\,\frac{f(x)}{x-x_i}\,dx + \right. \\
&\left. + \int_{x_i}^{x_{i+1}} \psi_i(y-a(x))\,\frac{f(x)}{x-x_{i+1}}\,dx\right) = \\
&= \sum_{i=1}^{n-2}\ \sum_{\odot\in\{\ominus,\cap,\oplus\}}\left(\varphi_i^{\odot}\int_{\omega_{a,i}^{\odot}(y)} \frac{f(x)}{x-x_i}\,dx + \psi_i^{\odot}\int_{\omega_{a,i}^{\odot}(y)} \frac{f(x)}{x-x_{i+1}}\,dx\right) = \\
&= \sum_{i=1}^{n-2} \varphi_i^{\cap}\left(\int_{\omega_{a,i}^{\cap}(y)} \frac{f(x)}{x-x_i}\,dx + \int_{\omega_{a,i}^{\ominus}(y)} \frac{f(x)}{x-x_i}\,dx\right) + \\
&+ \psi_i^{\cap}\left(\int_{\omega_{a,i}^{\cap}(y)} \frac{f(x)}{x-x_{i+1}}\,dx + \int_{\omega_{a,i}^{\oplus}(y)} \frac{f(x)}{x-x_{i+1}}\,dx\right) = \\
&= \sum_{i=1}^{n-2} \frac{1}{2\,|A_{i,i+1}|}\left(\int_{\omega_{a,i}^{-}(y)} \frac{f(x)}{x-x_i}\,dx + \int_{\omega_{a,i}^{+}(y)} \frac{f(x)}{x-x_{i+1}}\,dx\right)\ . \qquad (39)
\end{aligned}
$$

The last equality is met due to the relations

$$
\begin{aligned}
\omega_{a,i}^{\cap}(y) \cap \omega_{a,i}^{\ominus}(y) &= \varnothing\,, \\
\omega_{a,i}^{\cap}(y) \cap \omega_{a,i}^{\oplus}(y) &= \varnothing\,, \\
\omega_{a,i}^{-}(y) &= \omega_{a,i}^{\cap}(y) \cup \omega_{a,i}^{\ominus}(y)\,, \\
\omega_{a,i}^{+}(y) &= \omega_{a,i}^{\cap}(y) \cup \omega_{a,i}^{\oplus}(y)
\end{aligned}
$$

that can be easily proved.

The ranges of integration $\omega^{\square}_{a,i}(y)$, $i \in \{1\,, 2\,, \ldots\,, n-2\}$, $\square \in \{-\,,+\}$ fulfil equalities that are analogous to (27).

$$\omega^{\square}_{a,i}(y) = \{x \in (x_i\,,\, x_{i+1}) \,|\, y - a(x) \in (s^{\square}_i(x)\,,\, S^{\square}_i(x))\} \ . \tag{40}$$

With the aid of (39) formula (37) can be easily rewritten to get well-arranged form if $\omega^{+}_{a,0}(y)$ and $\omega^{-}_{a,n-1}(y)$ are defined by (40) and

$$\omega^{-}_{a,0}(y) = \omega^{+}_{a,0}(y) \ , \quad \omega^{+}_{a,n-1}(y) = \omega^{-}_{a,n-1}(y) \ .$$

The final form of the corrected pdf $\bar{f}_n$ then follows:

$$\bar{f}_n(y) = \sum_{i=0}^{n-1} \frac{1}{2\,|A_{i,i+1}|} \left(\int_{\omega^{-}_{a,i}(y)} \frac{f(x)}{x - x_i}\, dx + \int_{\omega^{+}_{a,i}(y)} \frac{f(x)}{x - x_{i+1}}\, dx \right) \ . \tag{41}$$

Now, in comparison with (26), we can see that this formula generalizes the previous case described in section 4.2.2.

The sequence of $\bar{f}_n$ converges to the exact distribution f_b when the number n increases. Usually a few points is sufficient as the example in Section 5 shows.

4.3 Computational aspects

Evaluation of the corrected pdf consists from two basic operations:

- integrations in (14) and (41),
- computation of the ranges of integration $\omega_a(y) \cap U_f$, resp. $\omega^{\square}_{a,i}(y) \cap U_f$ according to (13) and the remark below formula (7).

Integrations There are no problems with integration in (14) since the integral can be expressed in terms of cumulative distribution function (cdf) of pdf f. Cdf's of common probability distributions are available in the most of the statistical software packages.

To evaluate the integrals in (41), a special function

$$\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathbb{R} : [x\,,\, t] \mapsto \mathcal{F}(x,t) = \int \frac{f(x)}{x - t}\, dx$$

should be defined for a pdf f. Such a function could be programmed in advance and be applied similarly as cdf in the previous case.

In this way, integrations can be performed analytically.

Ranges of integration The bounding points of the sets $\omega_a(y)$, $\omega^{\square}_{a,i}(y)$ can be easily evaluated by means of solving pairs of simultaneous quadratic inequalities if the approximation function a is linear or quadratic. This feature is apparent from equations (18) and (40). Operation of intersection with set U_f does not require any additional computations.

Hence the algorithm for computation of the corrected pdf $\bar{f}$, resp. $\bar{f}_n$ is very effective since it needs not to use any iterative procedure.

5 Example

To illustrate the proposed procedure, a simplified version of the example from Subsection 2.2 will be computed through. Let the exact function be an one-dimensional projective transformation

$$b(x) = x' = \frac{-794.108\,x}{1.54891 + 0.655074\,x}\ ,$$

where x is a coordinate in the image and x' is the cartographic coordinate. The approximative function a is linear function

$$a(x) = -387.143\,x$$

computed from two control points ($x_1 = 0$ cm, $x'_1 = 0$ km, $x_2 = -0.70$ cm, $x'_2 = -271$ km). The probability distribution of X is normal distribution with mean $\mu = 0.2$ cm and standard deviation $\sigma = 0.02$ cm, which was truncated and normalized to have a constrained support $U_f = (0.12\,\text{cm}\,,\,0.28\,\text{cm})$. The pdf of the truncated normalized normal distribution is:

$$f(x) = \begin{cases} \dfrac{1}{\mathrm{erf}(|U_f|/\sqrt{8}\sigma)\sqrt{2\pi}\sigma} \exp\left(-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right) & \Leftarrow x \in U_f\,, \\ 0 & \Leftarrow x \notin U_f\,, \end{cases}$$

where $\mathrm{erf}(x)$ is error function defined by:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt\ .$$

Function b is evaluated in the bounding points of the support.

The corrected distribution was computed by means of formula (14). Its probability density function $\bar{f}$ is compared with other related density functions f_a, f_b in Fig. 4. The figure shows how the correction moves the approximative distribution to the exact distribution.

6 Conclusion

The proposed method enables safe and reliable use of an approximative function with random variable. The method is based on superposition of the uncertainty of the approximative function and the approximation error. The more complex knowledge of the approximation error is available, the better fit of the corrected and exact distributions is achieved. It is noteworthy that the corrected distribution is reliable irrespectively of the level of knowledge of the approximation error.

The algorithm for computation of the corrected distribution is very effective because it does not include any iterative procedures.

The suggested approach can be applied to a wide area of scientific and technical problems dealing with non-linear parameter estimation including Bayesian methods, nonlinear regression and other estimation techniques. Furthermore, it can help in designing the optimal level of approximation which ensures reliable decisions with minimal computational demands.

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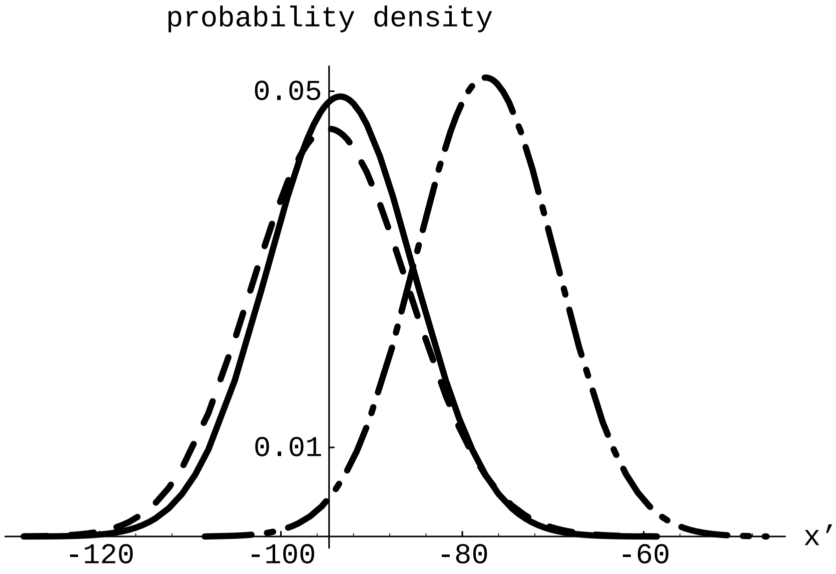

exact distribution f_b

approximative distribution f_a

corrected distribution $\bar{f}$

Figure 4: Probability distribution of perspective projection approximated by affinity