Shape Optimization of Elasto-Plastic Bodies Governed by Variational Inequalities*

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1 Introduction

The subject of this paper is shape optimization of elasto-perfectly plastic bodies obeying Hencky's law, in particular, numerical realization of this problem. As the state problem (of elasto-plasticity) can be formulated as a variational inequality (VI), the shape optimization problem leads to an interesting nonstandard control problem of the form

$$\begin{array}{rl} & g(\mathbf{u},\mathbf{y}) \to \inf \\ \text{subject to} & \\ & \mathbf{y} \text{ solves } \mathrm{VI}_{\mathbf{u}} \\ & \mathbf{u} \in \mathbf{U}, \end{array}$$

where $\mathbf{u}$ is the *control* variable, $\mathbf{y}$ the *state* variable, $\mathrm{VI}_{\mathbf{u}}$ is the state problem depending on the control $\mathbf{u}$ and $\mathbf{U}$ is a compact set. The main difficulty in solving such problems is that the map assigning to the control $\mathbf{u}$ the state $\mathbf{y}$ is generally only locally Lipschitz (under some assumptions which will be specified later), i.e., it is *nondifferentiable* (nonsmooth). The standard way of solving such problems is to use the so-called regularization technique: rewrite the VI as an optimization problem (if it is possible) and then penalize the constraints in this problem by a smooth penalty. This technique is widely used, e.g., in the book [5]. It is obvious that the solution obtained by means of the regularization technique may be far from the true solution of the original problem. It is the goal of this paper to solve such problems "directly", by solving the VI (or the corresponding optimization problem) without any inexact penalization. Then, of course, we have to take care of the nondifferentiability of the problem and use a nondifferentiable optimization (NDO) code to its numerical solution.

So far, we have solved two problems of this type by such a direct method, proposed and investigated in [10]. The analysis, results and comparison with

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regularization technique have been presented in [8, 10, 14]. However, the problem of shape optimization of elasto-plastic bodies is much more complicated and consequently much more difficult to solve as we shall see in the next sections.

2 Formulation of the Problem

We start with the formulation of the state problem of elasto-plasticity. Let $\Omega \subset \mathsf{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Assume that

$$\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_g, \quad \Gamma_u \cap \Gamma_g = \emptyset,$$

each of the part Γ_u and Γ_g being open in $\partial\Omega$.

Let $\mathsf{R}_s^{2\times 2}$ denote the set of symmetric 2×2 matrices. We introduce the set of *stress fields*

$$S(\Omega) = \{\tau : \Omega \to \mathsf{R}_s^{2\times 2} \;\big|\; \tau_{ij} \in L^2(\Omega), i, j = 1, 2\}$$

and the isomorphism

$$b : S(\Omega) \to S(\Omega)$$

defined by the generalized inverse Hooke's law. We consider an isotropic material and plain strain relations, so the relation $e = b\tau$ can be written as

$$\begin{aligned} e_{11} &= \frac{1+\nu}{E}[\tau_{11} - \nu(\tau_{11} + \tau_{22})] \\ e_{22} &= \frac{1+\nu}{E}[\tau_{22} - \nu(\tau_{11} + \tau_{22})] \\ e_{12} &= \frac{1+\nu}{E}\tau_{12}, \end{aligned}$$

where e is the *strain field*, E the Young modulus and ν the Poisson ratio, satisfying $E > 0$ and $0 \le \nu < \frac{1}{2}$. Denote by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ the $L^2(\Omega)$ inner product and norm, respectively, and by $((\cdot,\cdot))_\Omega$ and $|||\cdot|||_\Omega$ the energy inner product and norm, respectively, defined by

$$((\sigma,\tau))_\Omega = (b\sigma,\tau)_\Omega, \quad |||\sigma|||_\Omega = ((\sigma,\sigma))_\Omega^{1/2}.$$

From the definition of b it is easy to show that

$$b_0\|\sigma\|_\Omega \le |||\sigma|||_\Omega \le b_1\|\sigma\|_\Omega \quad \forall \sigma \in S(\Omega)$$

with some positive constants b_0, b_1.

Assume that a body force $F = (F_1, F_2) \in [L^2(\Omega)]^2$ and a surface traction $g = (g_1, g_2) \in [L^2(\Gamma_g)]^2$ are given. We define the set of *statically admissible* stress fields satisfying the *weak equilibrium conditions*

$$E(\Omega) = \{\tau \in S(\Omega) \;\big|\; (\tau, e(w))_\Omega = L_\Omega(w) \quad \forall w \in W(\Omega)\},$$

where

$$W(\Omega) = \{w = (w_1, w_2) \in [H^1(\Omega)]^2 \mid w = 0 \text{ on } \Gamma_u\},$$

$$(e(w))_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad i, j = 1, 2,$$

$$L_\Omega(w) = \sum_{i=1}^{2} \int_\Omega F_i w_i \,\mathrm{d}x + \sum_{i=1}^{2} \int_{\Gamma_g} g_i w_i \,\mathrm{d}s,$$

x_1, x_2 being the Cartesian coordinates. The vector field $w = (w_1, w_2)$ is called *displacement field.*

Assume that a yield function $\Upsilon : \mathsf{R}_s^{2\times 2} \to \mathsf{R}$ is given, which is convex, Lipschitz and such that

$$\Upsilon(\lambda\sigma) = |\lambda|\Upsilon(\sigma) \quad \forall \lambda \in \mathsf{R}, \forall \sigma \in \mathsf{R}_s^{2\times 2}.$$

In our considerations we shall use the von Mises function defined by

$$\Upsilon(\sigma) = \left(\sum_{i,j=1}^{2} \sigma_{ij}^D \sigma_{ij}^D + \frac{1}{9}(\sigma_{11} + \sigma_{22})^2\right)^{1/2},$$

where

$$\sigma_{ij}^D = \sigma_{ij} - \frac{1}{2}\delta_{ij}(\sigma_{11} + \sigma_{22}).$$

Furthemore, we define the set of *plastically admissible* stress fields

$$P(\Omega) = \{\tau \in S(\Omega) \;\big|\; \Upsilon(\tau) \le c_p \text{ a.e. in } \Omega\},$$

where c_p is a material constant.

The state variational inequality reads as

$$\begin{aligned} &\text{Find } \sigma \in E(\Omega) \cap P(\Omega) \text{ such that} \\ &((\sigma, \tau - \sigma))_\Omega \ge 0 \quad \forall \tau \in E(\Omega) \cap P(\Omega). \end{aligned}$$

It is well-known (cp., e.g., [3]) that this VI is equivalent to the following optimization problem

$$\begin{aligned} & \frac{1}{2}|||\tau|||_\Omega^2 \to \inf \\ \text{subject to} \quad & \\ & \tau \in E(\Omega) \cap P(\Omega), \end{aligned} \qquad (\mathbf{P}_\Omega)$$

known as *Haar-Kármán principle.*

Now, let us introduce the set of *admissible design variables*

$$U_{ad} = \{u \in C^{(0),1}([0,1]), \alpha \le u \le \beta, \left|\frac{\mathrm{d}u}{\mathrm{d}x_2}\right| \le c_1, \left|\frac{\mathrm{d}^2 u}{\mathrm{d}x_2^2}\right| \le c_2, \int_0^1 u \,\mathrm{d}x_2 = V\},$$

where α, β, c_1, c_2 and V are given positive numbers. Consider a family of admissible domains $\Omega(u)$ with variable right "vertical" part of the boundary

$$\Omega(u) = \{(x_1, x_2) \;\big|\; 0 < x_1 < u(x_2), 0 < x_2 < 1\}.$$

The shape optimization problem reads as

$$\left.\begin{array}{ll} & \mathcal{J}(\sigma(u)) \to \inf \\ \text{subject to} & \\ & \sigma(u) \text{ solves } (\mathbf{P}_{\Omega(u)}) \\ & u \in U_{ad}, \end{array}\right. \qquad (\mathcal{P})$$

where

$$\mathcal{J}(\sigma) = \int_{\Omega(u)} \Upsilon^2(\sigma(u))\,\mathrm{d}x.$$

That means, we allow the stresses to reach the plastic limit, but we require the overall stress to be as low as possible.

Now we discretize problems $(\mathbf{P}_\Omega)$ and $(\mathcal{P})$ by the finite element method. Let N be a positive number and $h = 1/N$. We denote by e_j, $j = 1, 2, \dots, N$, the subintervals $[(j-1)h, jh]$ and introduce the set

$$U_{ad}^h = \{u_h \in U_{ad} \mid u_h|_{e_j} \in P_1(e_j) \quad \forall j\},$$

where P_i denotes the set of polynomials of degree i. For $u_h \in U_{ad}^h$ let $\Omega(u_h)$ denote the polygonal computational domain. We construct the triangulation $\mathcal{T}(h, u_h)$ of $\Omega(u_h)$ by means of principal moving nodes (design nodes)

$$\hat{u}^j = (u_h(jh), jh), \quad j = 0, 1, \dots, N$$

and associated moving nodes, x_1-coordinates of which are given by an equidistant partition of the segments $[0, u_h(jh)]$. Thus for a fixed $h > 0$, the triangulation $\mathcal{T}(h, u_h)$ depends continuously on u_h. The domain $\Omega(u_h)$ with a given triangulation $\mathcal{T}(h, u_h)$ will be denoted by Ω_h.

Denoting the triangles of $\mathcal{T}(h, u_h)$ by K and using the convention that the symmetric 2×2 matrix will be written as a vector from R^3, we define the following finite element spaces

$$\begin{aligned} W_h(\Omega_h) &= \{w_h \in W(\Omega_h) \mid w_h|_K \in [P_1(K)]^2 \quad \forall K \in \mathcal{T}(h, u_h)\} \\ H_h(\Omega_h) &= \{\tau \in S(\Omega_h) \mid \tau|_K \in [P_0(K)]^3 \quad \forall K \in \mathcal{T}(h, u_h)\}, \end{aligned}$$

and the appropriate external approximation of the set $E(\Omega_h)$

$$E_h(\Omega_h) = \{\tau_h \in H_h(\Omega_h) \mid (\tau|_K, e(w_h))_{\Omega_h} = L(w_h) \quad \forall w_h \in W_h(\Omega_h)\}.$$

Instead of problems $(\mathbf{P}_\Omega)$ and $(\mathcal{P})$, we introduce the following approximate state problem

$$\left.\begin{array}{ll} & \frac{1}{2}|||\tau_h|||^2_{\Omega_h} \to \inf \\ \text{subject to} & \\ & \tau_h \in E_h(\Omega_h) \cap P(\Omega_h), \end{array}\right. \qquad (\mathbf{P}_{\Omega_h})$$

and the approximate shape optimization problem

$$\begin{array}{ll} & \mathcal{J}(\sigma_h(u_h)) \to \inf \\ \text{subject to} & \\ & \sigma_h(u_h) \text{ solves } (\mathbf{P}_{\Omega_h}) \\ & u_h \in U^h_{ad}. \end{array} \qquad (\mathcal{P}_h)$$

The question of the existence of solutions to $(\mathbf{P}_\Omega), (\mathcal{P}), (\mathbf{P}_{\Omega_h}), (\mathcal{P}_h)$, as well as the problem of convergence of the approximate solutions to the exact ones have been solved by Hlaváček [6].

Now, denote by $\mathbf{w} \in \mathsf{R}^{2n}$ the vector of nodal displacements, i.e.,

$$\mathbf{w}^{2i-1} = w_{h1}(\mathbf{n}_i),\ \mathbf{w}^{2i} = w_{h2}(\mathbf{n}_i), \quad i = 1, 2, \ldots, n,$$

where $\mathbf{n}_i$ are the nodes of the mesh, n is their number and $w_h = (w_{h1}, w_{h2})$ is the approximate displacement field. Similarly, denote by $\mathbf{y} \in \mathsf{R}^{3p}$ the vector of element stresses, i.e.,

$$\mathbf{y}^{3j-2} = \sigma_{h1}(K_j),\ \mathbf{y}^{3j-1} = \sigma_{h2}(K_j),\ \mathbf{y}^{3j} = \sigma_{h3}(K_j), \quad j = 1, 2, \ldots, p,$$

where K_j are the elements, p is their number and $\sigma_h = (\sigma_{h1}, \sigma_{h2}, \sigma_{h3})$ the approximate stress field. Denote further by $\mathbf{u} \in \mathsf{R}^N$ the vector of x_1-coordinates of the design nodes.

Denoting by $\boldsymbol{\varphi}_i = (\varphi_i, \varphi_i, \varphi_i)$ the basis functions of $H_h(\Omega_h)$, such that $\varphi_i = 1$ on K_i and is zero elsewhere, $i = 1, 2, \ldots, p$, we introduce the *flexibility* matrix $\mathbf{B}$, defined by

$$(\mathbf{B})^{ij} = ((\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j))_{\Omega_h}, \quad i, j = 1, 2, \ldots, p.$$

Similarly, denoting by $\boldsymbol{\psi}_i = (\psi_i, \psi_i)$ the (Courant) basis functions of $W_h(\Omega_h)$, we obtain the *equilibrium* matrix $\mathbf{A}$ and right-hand side vector $\mathbf{f}$, defined by

$$(\mathbf{A})^{ij} = (\boldsymbol{\varphi}_i, e(\boldsymbol{\psi}_j))_{\Omega_h}, \quad (\mathbf{f})^j = L_{\Omega_h}(\boldsymbol{\psi}_j), \quad i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, n.$$

Note that $\mathbf{B}$ is a block diagonal symmetric positive definite matrix of order $3p$ and $\mathbf{A}$ is a $3p \times 2n$ sparse matrix. Obviously, $\mathbf{B}, \mathbf{A}$ and $\mathbf{f}$ depend on the control vector $\mathbf{u}$, so we shall write $\mathbf{B}(\mathbf{u}), \mathbf{A}(\mathbf{u})$ and $\mathbf{f}(\mathbf{u})$ when necessary.

Finally, denote by $\mathbf{E}(\mathbf{u})$ the set isometrically isomorphic with $E_h(\Omega(u_h))$:

$$\mathbf{E}(\mathbf{u}) = \{\mathbf{v} \in \mathsf{R}^{3p} \;\big|\; \mathbf{A}(\mathbf{u})\mathbf{v} = \mathbf{f}(\mathbf{u})\};$$

by $\mathbf{P}$ the set

$$\mathbf{P} = \{\mathbf{v} \in \mathsf{R}^{3p} \;\big|\; \Upsilon^i(\mathbf{v}) \le 0, \quad i = 1, 2, \ldots, p\},$$

where

$$\Upsilon^i(\mathbf{v}) = \Upsilon(\mathbf{v}^{3i-2}, \mathbf{v}^{3i-1}, \mathbf{v}^{3i}) + c_p,$$

Υ being the von Mises function of plasticity; and by $\mathbf{U}_{ad}$ the set

$$\mathbf{U}_{ad} = \{\mathbf{u} \in \mathsf{R}^N \;\big|\; \mathbf{u}^i = u_h(jh) \text{ for some } u_h \in U^h_{ad}, j = 0, 1, \ldots, N\}.$$

Now, the discretized state problem $(\mathbf{P}_{\Omega_h})$ can be rewritten as a convex (non-quadratic) programming problem

$$\begin{aligned} & \frac{1}{2}\langle \mathbf{v}, \mathbf{B}(\mathbf{u})\mathbf{v}\rangle \to \inf \\ \text{subject to} \quad & \\ & \mathbf{v} \in \mathbf{E}(\mathbf{u}) \cap \mathbf{P} \end{aligned} \tag{2.1}$$

and the discretized optimal design problem $(\mathcal{P}_h)$ as

$$\begin{aligned} & \mathbf{J}(\mathbf{u}, \mathbf{y}) \to \inf \\ \text{subject to} \quad & \\ & \mathbf{y} \in \arg \min_{\mathbf{v}\in\mathbf{E}(\mathbf{u})\cap\mathbf{P}} \frac{1}{2}\langle \mathbf{v}, \mathbf{B}(\mathbf{u})\mathbf{v}\rangle \\ & \mathbf{u} \in \mathbf{U}_{ad} \end{aligned} \tag{2.2}$$

where

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{p} (\text{meas}(K_i)\Upsilon^i(\mathbf{v}))^2;$$

note that $\mathbf{J}(\mathbf{u}, \mathbf{v})$ depends explicitly on the control $\mathbf{u}$.

Denoting by $S(\mathbf{u})$ the map

$$S : \mathbf{u} \mapsto \arg \min_{\mathbf{v}\in\mathbf{E}(\mathbf{u})\cap\mathbf{P}} \frac{1}{2}\langle \mathbf{v}, \mathbf{B}(\mathbf{u})\mathbf{v}\rangle,$$

(2.2) may be rewritten into the form

$$\begin{aligned} & \Theta(\mathbf{u}) \equiv \mathbf{J}(\mathbf{u}, S(\mathbf{u})) \to \inf \\ \text{subject to} \quad & \\ & \mathbf{u} \in \mathbf{U}_{ad}. \end{aligned} \tag{2.3}$$

Assume that

(A1) the gradients $\bigtriangledown \Upsilon^i(\mathbf{y})$ for $i \in I(\mathbf{y}) \equiv \{j \in \{1, 2, \dots, p\} \mid \Upsilon^j(\mathbf{y}) = 0\}$ are linearly independent at any $\mathbf{y} \in S(\mathbf{u}), \mathbf{u} \in \mathbf{U}_{ad}$.

Due to this assumption and the positive definiteness of $\mathbf{B}(\mathbf{u})$ for all $\mathbf{u} \in \mathbf{U}_{ad}$, S is single-valued on $\mathbf{U}_{ad}$; moreover, it is locally Lipschitz ([2]) and directionally differentiable ([7, 9]). The same is true about the operators $\Lambda_1 : \mathbf{u} \mapsto \mu \in \mathsf{R}^{2n}$,$\Lambda_2 : \mathbf{u} \mapsto \lambda \in \mathsf{R}^p_+$ assigning to vectors $\mathbf{u}_0 \in \mathbf{U}_{ad}$ the multipliers μ_0, λ_0, corresponding to the equality and inequality constraints in the inner problem (2.1), respectively.

Hence, to numerical solution of (2.3), existing NDO methods can be applied, provided we are able to compute subgradients[1] from $\Theta(\mathbf{u})$ for arbitrary $\mathbf{u} \in \mathbf{U}_{ad}$. This is the subject of the next section.

[1]The term subgradient is borrowed from convex analysis, where it is used for the vectors belonging to the subdifferential. Recently, however, it is frequently used also in a nonconvex setting for the elements of various generalized gradients.

3 Computation of the Subgradients

Assume that at $\mathbf{u}_0 \in \mathbf{U}_{ad}$ the strict complementarity condition holds, i.e., for

$$I(\mathbf{u}_0) \equiv \left\{ i \in \{1, 2, \ldots, p\} \;\middle|\; \Upsilon^i(S(\mathbf{u}_0)) = 0 \right\}$$
$$J(\mathbf{u}_0) \equiv \left\{ i \in I(\mathbf{u}_0) \;\middle|\; \lambda_0^i > 0 \right\}$$

one has

$$I(\mathbf{u}_0) = J(\mathbf{u}_0).$$

Denote by $\alpha^i(\mathbf{u})$ the $i-$th row of $\mathbf{A}(\mathbf{u})$ and by $\beta^i(\mathbf{u})$ the $i-$th component of $\mathbf{f}(\mathbf{u})$. Then it is well-known ([4]) that S is even differentiable at $\mathbf{u}_0$ and $\bigtriangledown S(\mathbf{u}_0)$ is given as the operator which assigns to an arbitrary vector $\mathbf{z} \in \mathsf{R}^{3p}$ the (unique) solution of the quadratic program

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{Q}(\mathbf{u}_0)\mathbf{v}\rangle + \langle q(\mathbf{u}_0), \mathbf{v}\rangle \to \inf$$

subject to

$$\begin{aligned} &\langle \bigtriangledown \alpha^i(\mathbf{u}_0)\mathbf{z}, \mathbf{y}_0\rangle + \langle \bigtriangledown \beta^i(\mathbf{u}_0), \mathbf{z}\rangle + \langle \alpha^i(\mathbf{u}_0), \mathbf{v}\rangle = 0, \quad && i = 1, 2, \ldots, 2n, \\ &\langle \bigtriangledown \Upsilon^i(\mathbf{y}_0), \mathbf{v}\rangle = 0, && i \in I(\mathbf{u}_0), \end{aligned} \tag{3.1}$$

where $\mathbf{y}_0 = S(\mathbf{u}), \mu_0 = \Lambda_1(\mathbf{u}_0), \lambda_0 = \Lambda_2(\mathbf{u}_0), \mathbf{Q}(\mathbf{u}_0) = \mathbf{B}(\mathbf{u}_0) + \sum_{i=1}^{p} \lambda_0^i \bigtriangledown^2 \Upsilon^i(\mathbf{y}_0)$ and $q(\mathbf{u}_0) = \left[\bigtriangledown_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_0)\mathbf{y}_0) + \sum_{i=1}^{2n} \mu_0^i \bigtriangledown \alpha^i(\mathbf{u}_0)\right]\mathbf{z}$. If the strict complementarity at $\mathbf{u}_0 \in \mathbf{U}_{ad}$ is violated, we will assume that

(A2) there exists an index set $\widetilde{J}(\mathbf{u}_0)$and a direction $h \in \mathsf{R}^N$ for which one has
$$J(\mathbf{u}_0 + \vartheta h) = I(\mathbf{u}_0 + \vartheta h) = \widetilde{J}(\mathbf{u}_0)$$
for all $\vartheta > 0$ sufficiently small.

Under this hypothesis we have the following result.

Proposition 3.1 *Let $\mathbf{u}_0 \in \mathbf{U}_{ad}$ and $\widetilde{J}(\mathbf{u}_0)$ satisfies (A2) with a suitable direction $h \in \mathsf{R}^N$. Then the operator $P_{\widetilde{J}}(\mathbf{u}_0)$ which assigns to an arbitrary vector $\mathbf{z} \in \mathsf{R}^N$ the (unique) solution of the quadratic program (2.1) with $I(\mathbf{u}_0)$ being replaced by $\widetilde{J}(\mathbf{u}_0)$ belongs to the generalized Jacobian $\partial S(\mathbf{u}_0)$[2].*

Proof. Due to (A2), S is differentiable at $\mathbf{u}_0 + \vartheta h$ for all ϑ sufficiently small. Denote $\mathbf{D}(\mathbf{u}_0)$ the $2n \times N$ matrix, the rows of which are the vectors $(\bigtriangledown \alpha^i(\mathbf{u}_0))^T \mathbf{y}_0 + \bigtriangledown \beta^i(\mathbf{u}_0)$, $i = 1, 2, \ldots, 2n$. Furthermore,for the sake of notational simplicity, let $\mathbf{u}_\vartheta = \mathbf{u}_0 + \vartheta h$, $\mathbf{y}_\vartheta = S(\mathbf{u}_\vartheta)$, $\mu_\vartheta = \Lambda_1(\mathbf{u}_\vartheta)$ and $\lambda_\vartheta = \Lambda_2(\mathbf{u}_\vartheta)$. As it has been shown in [11],

$$\bigtriangledown S(\mathbf{u}_\vartheta) = \Xi_1(\mathbf{u}_\vartheta) \circ \left[-(\bigtriangledown_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_\vartheta)\mathbf{y}_\vartheta) - \sum_{i=1}^{2n} \mu_\vartheta^i \bigtriangledown \alpha^i(\mathbf{u}_\vartheta))\right] + \Xi_2(\mathbf{u}_\vartheta) \circ (-\mathbf{D}(\mathbf{u}_\vartheta)),$$

[2]See [1] for the definition.

where $\Xi_1(\mathbf{u}_\vartheta)$ projects $(\mathbf{Q}(\mathbf{u}_\vartheta))^{-1}\mathbf{b}, \mathbf{b} \in \mathsf{R}^{3p}$ onto the linear manifold

$$\{\mathbf{v} \in \mathsf{R}^{3p} \;\big|\; \langle \alpha^i(\mathbf{u}_\vartheta), \mathbf{v}\rangle = 0, i = 1, 2, \ldots, 2n, \langle \Upsilon^i(\mathbf{y}_\vartheta), \mathbf{v}\rangle = 0, i \in \widetilde{J}(\mathbf{u}_0)\}.$$

and $\Xi_2(\mathbf{u}_\vartheta)$ assigns to a vector $\mathbf{b}' \in \mathsf{R}^{2n}$ the minimum-norm element in the linear variety

$$\{\mathbf{v} \in \mathsf{R}^{3p} \;\big|\; \langle \alpha^i(\mathbf{u}_\vartheta), \mathbf{v}\rangle = \mathbf{b}'^i, i = 1, 2, \ldots, 2n\},$$

both in the $\mathbf{Q}(\mathbf{u}_\vartheta)-$metric. Evidently, for $\vartheta \downarrow 0$ one has $\mathbf{u}_\vartheta \to \mathbf{u}_0$, $\mathbf{y}_\vartheta \to \mathbf{y}_0$, $\mu_\vartheta \to \mu_0$ and $\lambda_\vartheta \to \lambda_0$, so that under the continuity assumptions being imposed

$$\lim_{\vartheta\downarrow 0} \bigtriangledown S(\mathbf{u}_\vartheta) = \Xi_1(\mathbf{u}_0) \circ \left[- \bigtriangledown_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_0)\mathbf{y}) - \sum_{i=1}^{2n} \mu_0^i \bigtriangledown \alpha^i(\mathbf{u}_0)\right] + \Xi_2(\mathbf{u}_0) \circ (-\mathbf{D}(\mathbf{u}_0))$$

which is exactly the operator $P_{\widetilde{J}}(\mathbf{u}_0)$. Thus, $P_{\widetilde{J}}(\mathbf{u}_0) \in \partial S(\mathbf{u}_0)$ by definition. $\square$

Of course, the verification of (A2) may be very difficult. In [8, 12] criteria have been derived to this purpose, but they also can hardly be applied in this complicated case. In particular, hypothesis (A2) is satisfied, if there exists such a perturbation h of the designed shape (specified by $\mathbf{u}_0$) that the values of the stress function decrease below the critical limit on all elements specified by indices $i \in I(\mathbf{u}_0) \setminus J(\mathbf{u}_0)$, whenever we replace the shape $\mathbf{u}_0$ by $\mathbf{u}_0 + \vartheta h$, ϑ being sufficiently small. However, in the computations performed we have not observed any difficulties due to "wrong" subgradients which indicates that the strict complementarity is rarely violated during the iteration process.

Thus, under (A2) (or under the strict complementarity) we are able to compute a matrix from $\partial S(\mathbf{u}_0)$ (or directly $\bigtriangledown S(\mathbf{u}_0)$), but the used NDO code requires a vector from $\partial\Theta(\mathbf{u}_0)$. However, this vector may be computed on the basis of Proposition 3.1 quite easily without an explicit knowledge of a matrix from $\partial S(\mathbf{u}_0)$.

Proposition 3.2 *Let* $\mathbf{u}_0 \in \mathbf{U}_{ad}$ *and* $\widetilde{J}(\mathbf{u}_0)$ *satisfy (A2) with a suitable direction* $h \in \mathsf{R}^N$. *Suppose that* $\mathbf{p}_1$ *is the (unique) solution of the adjoint quadratic program*

$$\begin{aligned} & \tfrac{1}{2}\langle \mathbf{p}_1, \mathbf{Q}(\mathbf{u}_0)\mathbf{p}_1\rangle - \langle \bigtriangledown_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0, \mathbf{y}_0), \mathbf{p}_1\rangle \to \inf \\ \textit{subject to} \quad & \\ & \langle \alpha^i(\mathbf{u}_0), \mathbf{p}_1\rangle = 0, \quad i = 1, 2, \ldots, 2n \\ & \langle \bigtriangledown \Upsilon^j(\mathbf{y}_0), \mathbf{p}_1\rangle = 0, \quad j \in \widetilde{J}(\mathbf{u}_0), \end{aligned} \tag{3.2}$$

and the vector $\mathbf{p}_2$ *is composed from the Lagrange multipliers of (3.2) at* $(\mathbf{u}_0, \mathbf{y}_0)$ *corresponding to the first* $2n$ *equality constraints. Then one has*

$$\xi \equiv \bigtriangledown_{\mathbf{u}}\mathcal{J}(\mathbf{u}_0, \mathbf{y}_0) - \left[\bigtriangledown_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_0)\mathbf{y}_0) + \sum_{i=1}^{2n} \mu_0^i \bigtriangledown \alpha^i(\mathbf{u}_0)\right]^T \mathbf{p}_1 - (\mathbf{D}(\mathbf{u}_0))^T\mathbf{p}_2 \in \partial\Theta(\mathbf{u}_0). \tag{3.3}$$

Proof. According to the Jacobian Chain Rule [1]

$$\nabla_{\mathbf{u}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0) + M^T \nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0) \in \partial\Theta(\mathbf{u}_0),$$

whenever $M \in \partial S(\mathbf{u}_0)$. Let $\mathbf{z} \in \mathsf{R}^N$ be an arbitrary vector. By using the expression for $P_{\tilde{J}}(\mathbf{u}_0)$ used in the proof of the preceding assertion, one obtains

$$\begin{aligned}
&\langle \nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0), P_{\tilde{J}}(\mathbf{u}_0)\mathbf{z}\rangle = \\
&\qquad \langle \nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0), \Xi_1(\mathbf{u}_0)\Big[-\nabla_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_0)\mathbf{y}_0) - \sum_{i=1}^{2n}\mu_0^i \nabla \alpha^i(\mathbf{u}_0)\Big]\mathbf{z} \\
&\qquad\qquad + \Xi_2(\mathbf{u}_0)(-\mathbf{D}(\mathbf{u}_0))\mathbf{z}\rangle = \\
&\qquad \langle \Big[-\nabla_{\mathbf{u}}(\mathbf{B}(\mathbf{u}_0)\mathbf{y}_0) - \sum_{i=1}^{2n}\mu_0^i \nabla \alpha^i(\mathbf{u}_0)\Big]^T \Xi_1(\mathbf{u}_0)\nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0), \mathbf{z}\rangle \\
&\qquad\qquad + \langle(-\mathbf{D}(\mathbf{u}_0))^T(\Xi_2(\mathbf{u}_0))^T \nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0), \mathbf{z}\rangle,
\end{aligned}$$

because the projector $\Xi_1(\mathbf{u}_0)$ is symmetric. As $\Xi_1(\mathbf{u}_0)\nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0) = \mathbf{p}_1$, and $(\Xi_2(\mathbf{u}_0))^T \nabla_{\mathbf{y}}\mathcal{J}(\mathbf{u}_0,\mathbf{y}_0) = \mathbf{p}_2$ due to Lemma 2.1 in [11], the assertion has been proved. □

4 Numerical Example

Consider the problem of optimal design of elasto-perfectly plastic body, in which the initial domain $\Omega_0 \equiv \Omega(\mathbf{u}_0)$ is depicted in Figure 1. Further, let $\Gamma_u = \{(x_1,x_2) \mid 0 \le x_1 \le 1.2, x_2 = 0\}$ be the part of the boundary $\partial\Omega_0$ with prescribed zero displacements, and let the constant surface traction $g = (1,0)$ is prescribed on the left vertical part of $\partial\Omega_0$, i.e., on the segment $\{(x_1,x_2) \mid x_1 = 0, 0 \le x_2 \le 1\}$. The surface traction on the rest of $\partial\Omega_0$ is prescribed as zero, as well as the body force F. The domain is discretized by triangles defined by 7×7 grid of nodes. The set of admissible controls $\mathbf{U}_{ad}$ is characterized by numbers

$$\alpha = 0.4,\ \beta = 1.6,\ c_1 = 5.0,\ c_2 = 4.0,\ V = 1.$$

Thus we have 216 state variables (72 elements $\times$ 3 unknowns per element) and 7 design variables (x_1−coordinates of principal moving nodes). The discretized state (nonlinear programming) problems (2.1) were solved by the SQP (Sequential Quadratic Programming) code NLPQLD due to K. Schittkowski [13], while the master optimization problems (2.3) by the NDO code BT (Bundle–Trust Region) due to H. Schramm and J. Zowe [14]; in particular, by its nonconvex version BTNCLC which can handle linear constraints. Figure 2 shows the optimal shape design obtained after 61 iterations of BTNCLC. The initial value of the objective functional was $\mathcal{J}_{INI} = 2.3153$ and the optimal value was $\mathcal{J}_{OPT} = 1.9500$ (i.e., 84% of $\mathcal{J}_{INI}$). In both figures, the elements where the plastic constraints were active — the so-called region of plasticity — are depicted by shading. We see that this region dramatically decreased after optimization.

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