Semigraphoids and structures of probabilistic conditional independence *

Milan Studený

Institute of Information Theory and Automation, Academy of Sciences of Czech Republic, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic
E-mail: studeny@utia.cas.cz

The concept of conditional independence (CI) has an important role in probabilistic reasoning, that is a branch of artificial intelligence where knowledge is modeled by means of a multidimensional finite-valued probability distribution. The structures of probabilistic CI are described by means of semigraphoids, that is lists of CI-statements closed under four concrete inference rules, which have at most two antecedents. It is known that every CI-model is a semigraphoid, but the converse is not true. In this paper, the semigraphoid closure of every couple of CI-statements is proved to be a CI-model. The substantial step to it is to show that every probabilistically sound inference rule for axiomatic characterization of CI properties (= axiom), having at most two antecedents, is a consequence of the semigraphoid inference rules. Moreover, all potential dominant triplets of the mentioned semigraphoid closure are found.

1. Introduction

Many reasoning tasks arising in uncertainty processing in artificial intelligence can be considerably simplified if a suitable concept of relevance or irrelevance of symptoms or variables into consideration is utilized. The conditional irrelevance in probabilistic reasoning is modeled by means of the concept of probabilistic *conditional independence* (CI) – for details see the book [12]. The fact that every CI-statement can be interpreted as certain qualitative relationship among random variables into consideration makes it possible to reduce the dimensionality of the problem and thus to find a more effective way of storing the knowledge base of a probabilistic expert system. This dimensionality reduction is important especially for the intensional approach to expert systems [14]. Note that the concept of CI has been introduced and studied also in nonprobabilistic calculi for dealing with uncertainty in artificial intelligence, for overview see [17] or [22]. Moreover probabilistic CI is also behind structural models in multivariate statistics [25]. The importance of the concept of CI for statistical estimates was highlighted in seventies by Dawid who formulated some basic

* This work was supported by the grant n. 275105 of the Academy of Sciences of Czech Republic, by the grant n. 201/94/0471 of the Grant Agency of Czech Republic and by the grant MŠMT n. VS96008.

properties of probabilistic CI [2]. These formal properties were independently treated and interpreted from a philosophical point of view by Spohn [19].

The structures of CI were at first described by means of graphs in literature. Two trends can be recognized – by means of undirected graphs (the concept of the Markov network – see [1,12]) and by means of directed acyclic graphs (the concepts of the probabilistic influence diagram [16,18] or Bayesian network [12]). Moreover, various types of hybrid graphs (involving both directed and undirected edges) were used for this purpose. For example, recursive causal graphs [7] or chain graphs [8]. Nevertheless, the graphical approaches cannot describe all possible probabilistic CI-structures. A natural way to remove this disadvantage is to describe CI-structures by means of so-called *independency models*, that is lists of CI-statements. But such an approach would be unnecessarily wide: owing to the above mentioned formal properties of CI [2,19] many independency models cannot be models of probabilistic CI-structures. Therefore Pearl and Paz [12,13] introduced the concept of *semigraphoid* (resp., graphoid) as an independency model closed under four (resp., five) concrete inference rules expressing the above mentioned properties of CI. Such an inference rule claims that if some CI-statements called *antecedents* are involved in a model of probabilistic CI-structure then another CI-statement called the *consequent* has to be involved in that model of CI-structure, too. Thus, every model of a probabilistic CI-structure is a semigraphoid and Pearl [12] conjectured the converse statement: that every semigraphoid is a CI-model (resp., every graphoid is a CI-model induced by a strictly positive probability distribution [13]).

However, this conjecture was refuted firstly by finding a further independent formal property of probabilistic CI [20] and later by showing that CI-models cannot be characterized as independency models closed under a finite number of inference rules [21]. This motivated an alternative approach to description of probabilistic CI-structures by means of so-called imsets developed in [23]. On the other hand, some important substructures of CI-structure have such a finite complete axiomatic characterization. Thus, semigraphoid inference rules are complete for the class of full CI-statements, that is CI-statements involving all variables into consideration [6,9], but also for the class of marginal CI-statements, that is CI-statements with fixed set of conditioning variables [4,11].

In this paper we present an additional result on probabilistic CI which at least partially justifies the concept of semigraphoid and shows that Pearl's original conjecture [12] was well-founded. Every probabilistically sound inference rule for CI-models, having at most two antecedents, is proved to be a consequence of the semigraphoid inference rules. This relative completeness of the semigraphoid inference rules makes it possible to show the main result of the paper declaring that the semigraphoid closure of every couple of CI-statements is a CI-model. Moreover, this semigraphoid closure is characterized in terms of its *dominant triplets* (newly introduced concept) what gives a simple method for its computer generating. Note that the results of this paper were already announced without proofs in the conference contribution [24], although with a little bit different terminology.

The structure of the paper can be described as follows. First, in the following (second) section basic concepts are recalled. Then, in the third section we introduce the concept of dominant triplet of a semigraphoid and give a list of candidates for dominant triplets of the semigraphoid closure of a couple of CI-statements. Using this list of candidates we obtain in the fourth section the list of conditions which will appear to characterize CI-statements outside the semigraphoid closure. To show that these CI-statements are not probabilistically implied by the given couple of CI-statements we use some special constructions of probability distributions in the fifth section. Finally, these constructions make it possible to derive almost immediately all desired results in the sixth section. The appendix (the seventh section) contains a technical proof of the only lemma from the fourth section.

2. Basic definitions and facts

Throughout the paper the symbol N denotes a finite nonempty set of *variables*. Every CI-statement over N will be described by means of a triplet $\langle A, B|C\rangle$ of pairwise disjoint subsets of N where A and B are nonempty. Here, the symbol $|$ is used to separate the third set of variables which has specific meaning of the conditioned area. The first set of variables and the second set of variables have meaning of independent areas. *Symmetric image* of a triplet $\langle A, B|C\rangle$ is the triplet $\langle B, A|C\rangle$.

The class of all such triplets $\langle A, B|C\rangle$ will be denoted by $\mathcal{T}(N)$. An *independency model* is simply a subset of $\mathcal{T}(N)$.

Convention 1. For sake of brevity we will often use the juxtaposition UV to denote the union $U \cup V$ of sets of variables $U, V \subset N$.

Definition 2 (conditional independence, probabilistic implication, CI-model). A probability distribution over N will be specified by a collection of nonempty finite sets $\{\mathbf{X}_i; i \in N\}$ and by a function

$$P: \prod_{i\in N} \mathbf{X}_i \to [0,1] \quad \text{with} \quad \sum \left\{ P(x);\ x \in \prod_{i\in N} \mathbf{X}_i \right\} = 1.$$

Whenever $\emptyset \neq A \subset N$ and P is a probability distribution over N its marginal distribution on A is a probability distribution P^A (over A) defined as follows ($P^N \equiv P$):

$$P^A(a) = \sum \left\{ P(a,b);\ b \in \prod_{i\in N\setminus A} \mathbf{X}_i \right\} \quad \text{for } a \in \prod_{i\in A} \mathbf{X}_i.$$

Having $\langle A, B|C\rangle \in \mathcal{T}(N)$ and a probability distribution P over N we will say that A is *conditionally independent of B given C with respect to P* or that the CI-statement

$\langle A,B|C\rangle$ holds for P and write $A \perp\!\!\!\perp B\,|\,C\,(P)$ if

$$\forall a \in \prod_{i\in A} \mathbf{X}_i \quad b \in \prod_{i\in B} \mathbf{X}_i \quad c \in \prod_{i\in C} \mathbf{X}_i$$
$$P^{ABC}(a,b,c)\cdot P^{C}(c) = P^{AC}(a,c)\cdot P^{BC}(b,c),$$

where we accept the convention $P^{\emptyset}(-) \equiv 1$.

Having $\{u_1,\dots,u_r\} \subset \mathcal{T}(N)$ $(r \geqslant 1)$ and $u_{r+1} \in \mathcal{T}(N)$ we will say that $\{u_1,\dots,u_r\}$ (*probabilistically*) *implies* u_{r+1} and write $\{u_1,\dots,u_r\} \models u_{r+1}$ if u_{r+1} holds for every probability distribution for which the CI-statements $u_1,\dots,u_r$ hold.

Finally, having a probability distribution P over N one can assign to it the independency model

$$\big\{\langle A,B|C\rangle \in \mathcal{T}(N);\ A \perp\!\!\!\perp B\,|\,C\,(P)\big\}$$

called the independency model induced by P. An independency model is called a (*consistent*) *CI-model over* N if it is induced by some probability distribution over N.

CI-models have the following important property – for proof see [21] or [5].

Lemma 3. The intersection of two CI-models over N (that is set of CI-statements belonging to both CI-models) is also a CI-model.

As mentioned in the Introduction the purpose of the presented approach is to describe CI-models in terms of so-called inference rules.

Definition 4 (inference rule, antecedent, consequent, probabilistic soundness). By an *inference rule* with r antecedents $(r \geqslant 1)$ we will understand an $(r+1)$-nary relation on $\mathcal{T}(N)$. We will say that an independency model $\mathcal{M} \subset \mathcal{T}(N)$ is *closed under* an inference rule $\mathcal{R}$ if for each *instance* of $\mathcal{R}$ (that is for every $(r+1)$-tuple $[u_1,\dots,u_{r+1}]$ of elements of $\mathcal{T}(N)$ belonging to $\mathcal{R}$) the *consequent* (that is u_{r+1}) belongs to $\mathcal{M}$ whenever the *antecedents* (that is $u_1,\dots,u_r$) belong to $\mathcal{M}$.

An inference rule $\mathcal{R}$ with r antecedents is called (*probabilistically*) *sound* if for every instance $[u_1,\dots,u_{r+1}]$ of $\mathcal{R}$ it holds $\{u_1,\dots,u_r\} \models u_{r+1}$.

Note that it can be proved using the property mentioned in Lemma 3 (see Proposition 2 in [21]) that CI-models can be potentially characterized as independency models closed under a countable number of sound inference rules (the characterization is understood to be shared among all possible finite sets of variables N).

Definition 5 (semigraphoid, derivability, semigraphoid closure). Usually, an inference rule is expressed by an informal schema, listing firstly antecedents and after an arrow the consequent. Thus, the following schemata:

$\langle A,B|C\rangle \longrightarrow \langle B,A|C\rangle$ symmetry
$\langle A,BC|D\rangle \longrightarrow \langle A,C|D\rangle$ decomposition

$\langle A, BC|D\rangle \longrightarrow \langle A, B|CD\rangle$ weak union
$[\langle A, B|CD\rangle$ & $\langle A, C|D\rangle] \longrightarrow \langle A, BC|D\rangle$ contraction

describe four inference rules. Every independency model closed under these inference rules will be called a *semigraphoid.*

Moreover, we will say that $u_{r+1} \in \mathcal{T}(N)$ is *derivable from* $\{u_1, \ldots, u_r\} \subset \mathcal{T}(N)$ ($r \geqslant 1$) and write $\{u_1, \ldots, u_r\} \vdash_{sem} u_{r+1}$ if there exists a derivation sequence $t_1, \ldots, t_n \subset \mathcal{T}(N)$ where $t_n = u_{r+1}$ and for each t_i ($i \leqslant n$) either $t_i \in \{u_1, \ldots, u_r\}$ or t_i is a direct consequence of some preceding t_js by virtue of some of above mentioned semigraphoid inference rules.

Having an independency model $\mathcal{M} \subset \mathcal{T}(N)$ its *semigraphoid closure* consists of all elements of $\mathcal{T}(N)$ derivable from $\mathcal{M}$ (it is evidently a semigraphoid).

As every semigraphoid inference rule is probabilistically sound owing to the formal properties of probabilistic CI from [2,19] one can easily see:

Lemma 6. Whenever $u_1, \ldots, u_{r+1} \in \mathcal{T}(N)$ then $\{u_1, \ldots, u_r\} \vdash_{sem} u_{r+1}$ implies

$$\{u_1, \ldots, u_r\} \models u_{r+1}.$$

Remark. The restrictions put on triplets of sets of variables describing CI-statements in the beginning of this section are not necessary in general, and some authors do not any restriction [2,3]. However, I have specific reasons for the limitation here. The first reason is that one has $A \perp\!\!\!\perp B \,|\, C\,(P)$ iff $(A \setminus C) \perp\!\!\!\perp (B \setminus C) \,|\, C\,(P)$ for arbirary (nondisjoint) triplet of variable sets A, B, C and any probability distribution P. Especially, every triplet, whose third component intersects either with the first component or with the second component, is superfluous from the point of view of description of CI-structure.

The second reason is that the statement $A \perp\!\!\!\perp A \,|\, C\,(P)$ has quite reasonable interpretation: the set of variables A is functionally dependent on the set of variables C in the distribution P, see [10]. However, the topic of this paper are pure CI-statements, not the statements about functional dependence!

Therefore I have decided to consider only the triplets of disjoint sets. Note that limitation of similar type is always necessary if one wants to avoid functional dependence statements. For example, the limitation $A \cap B \subset C$ to the triplets of variable sets A, B, C was used in the theory of relational databases [15], where the concept of embedded multivalued dependency is an analogue of the concept of CI.

The other restriction has also natural justification. One has always $A \perp\!\!\!\perp \emptyset \,|\, C\,(P)$ and $\emptyset \perp\!\!\!\perp B \,|\, C\,(P)$. Therefore, it has no sense to keep in mind such 'noninformative' trivial triplets. Moreover, the formal assumption that first two sets in triplets are nonempty is indeed utilized in the proofs below.

On the other hand, the third components of triplets have to be allowed to be empty. The reason is that the CI-statement $A \perp\!\!\!\perp B \,|\, \emptyset\,(P)$ means nothing but that A is independent of B in P unconditionally. Thus, any CI-model describes also the classic independence structure.

3. Dominant triplets

Our first actual step is to introduce a special ordering on $\mathcal{T}(N)$.

Definition 7 (dominant triplets). Supposing $\langle R,S|T\rangle$, $\langle E,F|G\rangle \in \mathcal{T}(N)$ we will say that $\langle E,F|G\rangle$ *dominates* $\langle R,S|T\rangle$ and write

$$\langle R,S|T\rangle \prec \langle E,F|G\rangle \quad \text{if} \quad \{R \subset E\} \;\&\; \{S \subset F\} \;\&\; \{T \supset G\} \;\&\; \{RST \subset EFG\}.$$

Having a semigraphoid $\mathcal{M} \subset \mathcal{T}(N)$ the maximal elements of $\mathcal{M}$ with respect to the ordering $\prec$ are called its *dominant* triplets.

Evidently, if $u \prec v$, then u can be derived from v by means of symmetry, decomposition and weak union. It is not difficult to see that the semigraphoid closure of one triplet $u \in \mathcal{T}(N)$ consists of all elements of $\mathcal{T}(N)$ dominated either by u or by its symmetric image.

In the rest of the paper we will often consider the following special situation.

Convention 8. Let three elements of $\mathcal{T}(N)$ denoted by $\langle A,B|C\rangle$, $\langle I,J|K\rangle$ and $\langle X,Y|Z\rangle$ are considered. Then we will keep the following notation for complementary sets: $D = N \setminus ABC$, $L = N \setminus IJK$, $W = N \setminus XYZ$.

Moreover, we will consider the group of 16 permutations generated by four elementary transposition $A \leftrightarrow B$, $I \leftrightarrow J$, $[A,B,C,D] \leftrightarrow [I,J,K,L]$, $X \leftrightarrow Y$. Thus, every permutation can transform any condition concerning relationship among sets A, B, C, D, I, J, K, L, X, Y, Z, W into an isomorphic situation. For example, the condition

$$\langle X,Y|Z\rangle \prec \langle A,B|C\rangle \Leftrightarrow [X \subset A \;\&\; Y \subset B \;\&\; Z \supset C \;\&\; W \supset D]$$

is transformed by $[A,B,C,D] \leftrightarrow [I,J,K,L]$ into

$$[X \subset I \;\&\; Y \subset J \;\&\; Z \supset K \;\&\; W \supset L] \Leftrightarrow \langle X,Y|Z\rangle \prec \langle I,J|K\rangle.$$

Analogously, arbitrary triplets expressed by means of those sets can be transformed into isomorphic triplets.

As a byproduct of this paper all potential dominant triplets of the semigraphoid closure of a couple of triplets in $\mathcal{T}(N)$ will be found. In fact, the candidates can be divided into four types described in the following lemma.

Lemma 9. Supposing $\langle A,B|C\rangle$, $\langle I,J|K\rangle$, $\langle X,Y|Z\rangle \in \mathcal{T}(N)$ let us consider the following conditions:

[1.] $\langle X,Y|Z\rangle \prec \langle A,B|C\rangle$,
[2.] $\langle X,Y|Z\rangle \prec \langle A \cap I, (J \setminus C) \cup (B \setminus L) \,|\, C \cup (A \cap K)\rangle$,
[3.] $\langle X,Y|Z\rangle \prec \langle A \cap I, B \cup (A \cap J) \,|\, C \cup (A \cap K)\rangle$,
[4.] $\langle X,Y|Z\rangle \prec \langle (A \cap I) \cup (B \cap J), (A \cap J) \cup (B \cap I) \,|\, CK\rangle$.

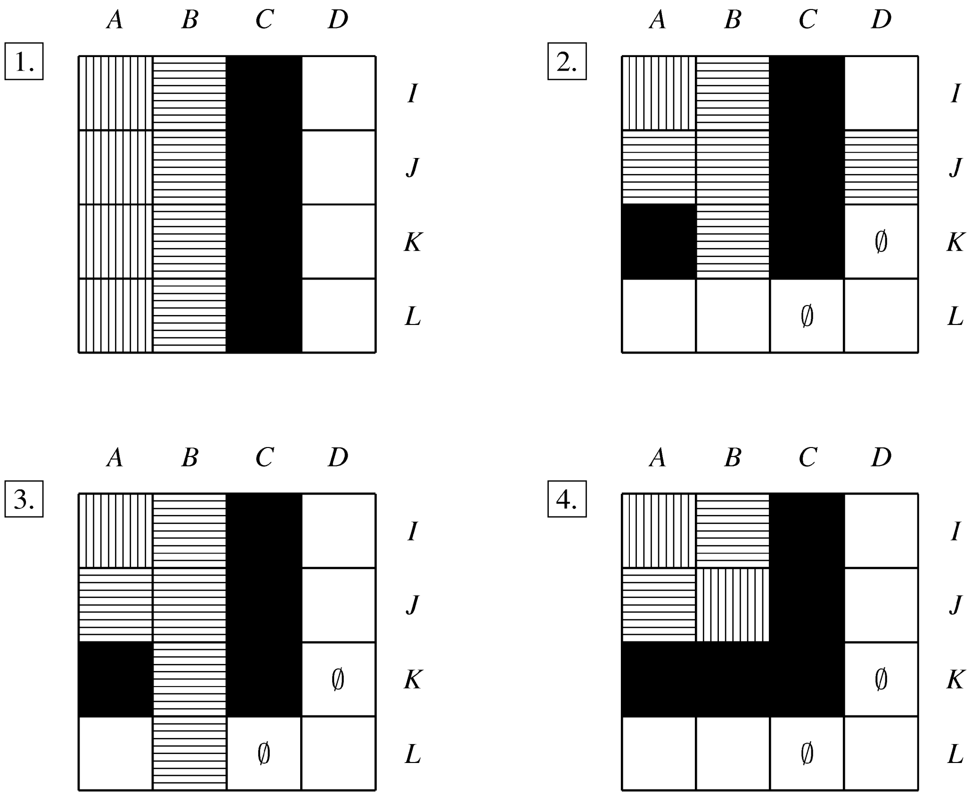

Figure 1. Each of these pictures describes informally one of the conditions from Lemma 9.

If $C \cap L = \emptyset = D \cap K$ and a situation isomorphic to one of the preceding four conditions occurs, then $\langle X, Y|Z\rangle$ belongs to the semigraphoid closure of $\{\langle A, B|C\rangle, \langle I, J|K\rangle\}$.

If the assumption $C \cap L = \emptyset = D \cap K$ is released, then that conclusion holds at least in the situations isomorphic to the first condition.

Remarks. 1) Of course, the conditions [2.]–[4.] automatically involve the requirement that the 'dominating' triplet indeed belongs to $\mathcal{T}(N)$.

2) The first type (that is the situations isomorphic to the first condition) represents 4 potential dominant triplets $\langle A, B|C\rangle$, $\langle B, A|C\rangle$, $\langle I, J|K\rangle$, $\langle J, I|K\rangle$. The second (resp., third) type represents 16 triplets, the fourth type 2 triplets only. Altogether it gives 38 potential dominant triplets. I use adjective 'potential' since in general the triplets can coincide or may be dominated by other triplets (see Example 1), but also may not be elements of $\mathcal{T}(N)$ (for example, if $A \cap I = \emptyset$ the 'dominating' triplet in [2.] has the first component empty).

3) The pictures from figure 1 informally illustrate the conditions from Lemma 9 by depicting the respective potential dominant triplets mentioned in the lemma. Note that the pictures describing isomorphic potential dominant triplets are omitted. Every

picture in nothing but Venn diagram, where the black area stands for the set of variables conditioned on, the white area stands for the uncovered set of variables, and where the independent areas are shown by vertical and horizontal hatching, respectively. The symbol for empty set indicates that the corresponding area is supposed to be empty.

Proof. Clearly, the claims for isomorphic situations can be obtained by the corresponding permutation. To show that $u = \langle A \cap I, (J \setminus C) \cup (B \cap IK) | C \cup (A \cap K) \rangle$ from [2.] belongs to the semigraphoid closure of $\{\langle A, B|C \rangle, \langle I, J|K \rangle\}$ one can realize that $\langle A \cap I, (J \setminus C) | (B \cap IK) \cup C \cup (A \cap K) \rangle$ is dominated by $\langle I, J|K \rangle$ (one uses implicitly the assumption $C \cap L = \emptyset = D \cap K$ here) and $\langle A \cap I, (B \cap IK) | C \cup (A \cap K) \rangle \prec \langle A, B|C \rangle$ and then use contraction to derive u.

To get the same conclusion for $v = \langle A \cap I, B \cup (A \cap J) | C \cup (A \cap K) \rangle$ from [3.] realize that $\langle A \cap I, B \cap L | (A \cap J) \cup (B \setminus L) \cup C \cup (A \cap K) \rangle \prec \langle A, B|C \rangle$ and combine it by virtue of contraction with $\langle A \cap I, (A \cap J) \cup (B \setminus L) | C \cup (A \cap K) \rangle$ which is dominated by the triplet u from [2.].

Finally, the triplet $\langle (A \cap I) \cup (B \cap J), A \cap J | (B \cap I) \cup C \cup K \rangle$ is dominated by the triplet $\langle (I \setminus C) \cup (B \setminus L), A \cap J | C \cup (A \cap K) \rangle$ (use $D \cap K = \emptyset$) whose symmetric image is isomorphic to the triplet from [2.] (use the transposition $I \leftrightarrow J$) and the triplet $\langle (A \cap I) \cup (B \cap J), B \cap I | C \cup K \rangle$ is dominated by the triplet $\langle (J \setminus C) \cup (A \setminus L), B \cap I | C \cup (B \cap K) \rangle$ (again use $D \cap K = \emptyset$) whose symmetric image is also isomorphic to the triplet from [2.] (use the transposition $A \leftrightarrow B$). Then, using the contraction inference rule one can derive the triplet from [4.]. □

Example 1. Every semigraphoid can be recorded shortly by the list of its dominant triplets. For example, consider the set of variables $N = 123456$ (for brevity, juxtaposition of elements indicates the set throughout this example) and the semigraphoid closure $\mathcal{M}$ of a couple of triplets $\langle A, B|C \rangle = \langle 12, 34|5 \rangle$, $\langle I, J|K \rangle = \langle 15, 236|\emptyset \rangle$. Lemma 9 gives the list of 38 potential dominant triplets of $\mathcal{M}$ (the completeness of that list will be proved later – see Corollary 15). However, the triplet $\langle A \cap I, (J \setminus C) \cup (B \setminus L) | C \cup (A \cap K) \rangle = \langle 1, 236|5 \rangle$ from [2.] is dominated by $\langle I, J|K \rangle$ and therefore it is not a dominant triplet of $\mathcal{M}$. On the other hand, its isomorphic triplet $\langle A \cap I, (B \setminus K) \cup (J \setminus D) | K \cup (C \cap I) \rangle = \langle 1, 234|5 \rangle$ (use $[A, B, C, D] \leftrightarrow [I, J, K, L]$) is not dominated by any other potential dominant triplet of $\mathcal{M}$. Similarly the triplets $\langle A \cap J, (B \setminus K) \cup (I \setminus D) | K \cup (C \cap J) \rangle = \langle 2, 1345|\emptyset \rangle$ and $\langle B \cap J, (A \setminus K) \cup (I \setminus D) | K \cup (C \cap J) \rangle = \langle 3, 125|\emptyset \rangle$ and of course $\langle A, B|C \rangle$, $\langle I, J|K \rangle$. The reader can verify that all other potential dominant triplets (except the symmetric images of 5 mentioned ones) are dominated by the mentioned five triplets or by their symmetric images. Thus, $\mathcal{M}$ has exactly ten dominant triplets.

4. Triplets outside the semigraphoid closure

In the preceding section some triplets belonging potentially to the semigraphoid closure of a couple of triplets $\langle A, B|C \rangle, \langle I, J|K \rangle$ were found. To show that they form a complete list of potential dominant triplets of the semigraphoid closure we need to characterize the triplets which are not dominated by any of the candidates for dominant triplets. These 'nondominated' triplets are explicitly characterized in the following pretty technical lemma.

Lemma 10. Supposing that $\langle A,B|C\rangle$, $\langle I,J|K\rangle$, $\langle X,Y|Z\rangle \in \mathcal{T}(N)$ one of the following three cases occurs:

(i) Either $C \cap L \neq \emptyset$ or $D \cap K \neq \emptyset$ and moreover $\langle X,Y|Z\rangle$ is dominated either by $\langle A,B|C\rangle$ or by $\langle I,J|K\rangle$ or by their symmetric images.

(ii) It holds $C \cap L = \emptyset = D \cap K$ and one of the situations isomorphic to the conditions from Lemma 9 occurs.

(iii) An isomorphic situation to one of 18 conditions listed below occurs.

The list of conditions:

(A.1) $XY \cap CD \cap KL \neq \emptyset$.
(A.2) $X \cap A \cap KL \neq \emptyset$ & $Y \cap ACD \neq \emptyset$.
(A.3) $X \cap A \cap I \neq \emptyset$ & $Y \cap A \cap I \neq \emptyset$.
(B.1) $Z \cap D \cap L \neq \emptyset$.
(B.2) $X \cap D \neq \emptyset$ & $XZ \cap L \neq \emptyset$.
(B.3) $X \cap CD \neq \emptyset$ & $Y \cap CD \neq \emptyset$ & $Z \cap L \neq \emptyset$.
(B.4) $X \cap ACD \neq \emptyset$ & $Y \cap ACD \neq \emptyset$ & $Z \cap ACD \cap L \neq \emptyset$.
(C.1) $W \cap C \cap K \neq \emptyset$.
(C.2) $X \cap C \neq \emptyset$ & $XW \cap K \neq \emptyset$.
(C.3) $X \cap CD \neq \emptyset$ & $Y \cap CD \neq \emptyset$ & $W \cap K \neq \emptyset$.
(C.4) $X \cap ACD \neq \emptyset$ & $Y \cap ACD \neq \emptyset$ & $W \cap ACD \cap K \neq \emptyset$.
(D) $Z \cap D \neq \emptyset$ & $Z \cap L \neq \emptyset$.
(E) $W \cap C \neq \emptyset$ & $W \cap K \neq \emptyset$.
(F) $Z \cap D \neq \emptyset$ & $W \cap D \cap K \neq \emptyset$.
(G) $Z \cap C \cap L \neq \emptyset$ & $W \cap C \neq \emptyset$.
(H) $X \cap ACD \neq \emptyset$ & $Y \cap IKL \neq \emptyset$ & $Z \cap D \cap I \neq \emptyset$ & $W \cap A \cap K \neq \emptyset$.
(I) $Z \cap D \cap I \neq \emptyset$ & $Z \cap D \cap J \neq \emptyset$ & $W \cap K \neq \emptyset$.
(J) $Z \cap D \neq \emptyset$ & $W \cap A \cap K \neq \emptyset$ & $W \cap B \cap K \neq \emptyset$.

Remarks. 1) Each condition from the list above can be transformed by one of considered permutations into an isomorphic situation, sometimes into itself. Thus, every condition describes a type with different number of represented situations. The types (A.1), (B.1), (C.1), (D), (E) involve just one situation, (B.3), (C.3), (F), (G), (I), (J) two situations, (A.3), (B.2), (B.4), (C.2), (C.4) four situations, (A.2) eight situations and (H) sixteen situations. Altogether 61 considered situations.

2) The conditions (i)–(iii) from Lemma 10 are in fact mutually exclusive. It is an easy consequence of later results (Lemma 12 together with Lemma 9). However, in

the proof of Lemma 10 we show only that they together exhaust the general situation (exactly what is needed).

The proof of Lemma 10 is technical and it is shifted to an appendix. In fact, it is a big 'decision tree' whose final leaves are either the situations isomorphic to four conditions from Lemma 9 or the situations isomorphic to 18 conditions above.

5. Constructions of probability distributions

In this section ten special constructions of probability distributions are given. They make it possible to complete the proof that the triplets outside the semigraphoid closure of a couple of CI-statements are not probabilistically implied by that couple.

Every of ten basic constructions $\boxed{\text{A}}$ – $\boxed{\text{J}}$ listed below gives a probability distribution over at most five-element set of variables. More concretely, we consider either the variable set $\{p,q\}$ or $\{p,q,r\}$ or $\{p,q,r,s\}$ or $\{p,q,r,s,t\}$. The conditions listed after Lemma 10 were already grouped and denoted in such a way that it is clear which basic construction will be needed to handle the respective condition: the condition (A.1)–(A.3) can be managed with help of the basic construction $\boxed{\text{A}}$, the conditions (B.1)–(B.4) using the construction $\boxed{\text{B}}$, etc.

One of possible construction follows every of the assertions $\boxed{\text{A}}$ – $\boxed{\text{J}}$ below. It is left to the reader to verify that it satisfies the requirements.

$\boxed{\text{A}}$ There exists a probability distribution Q over $\{p,q\}$ such that $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \{0,1\}$ and Q uniformly distributed on two configurations: $(0,0)$ and $(1,1)$. That means, Q assigns $1/2$ to the configurations $(0,0)$ and $(1,1)$ and 0 to the remaining configurations $(0,1)$ and $(1,0)$. □

$\boxed{\text{B}}$ There exists a probability distribution Q over $\{p,q,r\}$ such that $\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)$, $\{p\} \perp\!\!\!\perp \{r\} \,|\, \emptyset\,(Q)$, $\{q\} \perp\!\!\!\perp \{r\} \,|\, \emptyset\,(Q)$ and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)]$.
Especially, one has $\neg[\{p,r\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \{0,1\}$ and Q uniformly distributed on the following four configurations: $(0,0,0)$ $(0,1,1)$ $(1,0,1)$ $(1,1,0)$. □

$\boxed{\text{C}}$ There exists a probability distribution Q over $\{p,q,r\}$ such that $\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)$, $\{p\} \perp\!\!\!\perp \{r\} \,|\, \{q\}\,(Q)$, $\{q\} \perp\!\!\!\perp \{r\} \,|\, \{p\}\,(Q)$ and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$.
Especially, one has $\neg[\{p,r\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \{0,1\}$ and Q uniformly distributed on the following two configurations: $(0,0,0)$ and $(1,1,1)$. □

Convention 11. Supposing that $H \subset N$ and $e, f, g \in N \setminus H$ are distinct, the symbol $\{e\} \perp\!\!\!\perp \{f\} \perp\!\!\!\perp \{g\} \,|\, H\ (Q)$ denotes the complete joint independence of variables e, f, g with given H. It can be equivalently defined by listing all CI-statements $R \perp\!\!\!\perp S \,|\, T\,(Q)$ where $\langle R, S|T\rangle \in \mathcal{T}(N)$, $RST \subset \{e, f, g\} \cup H$ and $T \supset H$.

D There exists a probability distribution Q over $\{p, q, r, s\}$ such that
$\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{r\} \,|\, \emptyset\,(Q)$, $\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{s\} \,|\, \emptyset\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r, s\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \mathbf{X}_s = \{0, 1\}$ and Q uniformly distributed on the following eight configurations:
(0, 0, 0, 0) (0, 0, 1, 1) (0, 1, 0, 1) (0, 1, 1, 0)
(1, 0, 0, 1) (1, 0, 1, 0) (1, 1, 0, 0) (1, 1, 1, 1). □

E There exists a probability distribution Q over $\{p, q, r, s\}$ such that
$\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{r\} \,|\, \{s\}\,(Q)$, $\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{s\} \,|\, \{r\}\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \mathbf{X}_s = \{0, 1\}$ and Q uniformly distributed on two configurations: (0, 0, 0, 0) and (1, 1, 1, 1). □

F There exists a probability distribution Q over $\{p, q, r, s\}$ such that
$\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\ (Q)$, $\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{r\} \,|\, \{s\}\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \{0, 1\}$, $\mathbf{X}_s = \{0, 1, 2, 3\}$ and Q uniformly distributed on four configurations (the order of variables is p, q, r, s):
(0, 0, 0, 0) (0, 1, 1, 1) (1, 0, 1, 2) (1, 1, 0, 3). □

G There exists a probability distribution Q over $\{p, q, r, s\}$ such that
$\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r, s\}\,(Q)$, $\{p\} \perp\!\!\!\perp \{q\} \perp\!\!\!\perp \{s\} \,|\, \emptyset\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_s = \{0, 1\}$, $\mathbf{X}_r = \{0, 1, 2, 3\}$ and Q uniformly distributed on the following eight configurations (the order of variables is p, q, r, s):
(0, 0, 0, 0) (0, 0, 1, 1) (0, 1, 2, 0) (0, 1, 3, 1)
(1, 0, 3, 0) (1, 0, 2, 1) (1, 1, 1, 0) (1, 1, 0, 1). □

H There exists a probability distribution Q over $\{p, q, r, s\}$ such that
$\{p\} \perp\!\!\!\perp \{r, q\} \,|\, \{s\}\,(Q)$, $\{p, s\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \mathbf{X}_s = \{0, 1\}$ and define Q uniformly distributed on four configurations (the order of variables is p, q, r, s):
(0, 0, 0, 0) (0, 1, 1, 0) (1, 1, 0, 1) (1, 0, 1, 1). □

[I] There exists a probability distribution Q over $\{p,q,r,s,t\}$ such that
$\{p,r\} \perp\!\!\!\perp \{q,s\} \,|\, \{t\}\,(Q)$, $\{p,q\} \perp\!\!\!\perp \{t\} \,|\, \emptyset\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r,s\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \mathbf{X}_s = \mathbf{X}_t = \{0,1\}$ and Q uniformly distributed on the following eight configurations (the order of variables is p,q,r,s,t):
$(0,0,0,0,0)$ $(0,1,0,1,0)$ $(1,0,1,0,0)$ $(1,1,1,1,0)$
$(0,0,1,1,1)$ $(0,1,1,0,1)$ $(1,0,0,1,1)$ $(1,1,0,0,1)$. □

[J] There exists a probability distribution Q over $\{p,q,r,s,t\}$ such that
$\{p,s\} \perp\!\!\!\perp \{q,t\} \,|\, \emptyset\,(Q)$, $\{r\} \perp\!\!\!\perp \{p,q\} \,|\, \{s,t\}\,(Q)$
and $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \{r\}\,(Q)]$.

Indeed: Take $\mathbf{X}_p = \mathbf{X}_q = \mathbf{X}_r = \mathbf{X}_s = \mathbf{X}_t = \{0,1\}$ and Q uniformly distributed on the following four configurations (the order of variables is p,q,r,s,t):
$(0,0,0,0,0)$ $(0,1,1,0,1)$ $(1,0,1,1,0)$ $(1,1,0,1,1)$. □

Lemma 12. Suppose that $\langle A,B|C\rangle$, $\langle I,J|K\rangle$, $\langle X,Y|Z\rangle \in \mathcal{T}(N)$ and the case (iii) from Lemma 10 occurs. Then $\{\langle A,B|C\rangle, \langle I,J|K\rangle\} \not\models \langle X,Y|Z\rangle$.

Proof. Clearly, owing to permutations from Convention 8 it suffices to prove the desired conclusion whenever one of the conditions (A.1)–(J) occurs. Thus, one has to find a probability distribution P over N such that $A \perp\!\!\!\perp B \,|\, C\,(P)$, $I \perp\!\!\!\perp J \,|\, K\,(P)$ but $\neg[X \perp\!\!\!\perp Y \,|\, Z\,(P)]$. The idea is to choose (under each of the mentioned conditions) at most five-element subset S of N and construct a probability distribution Q over S according to one of ten basic constructions [A] – [J] listed above. Then one has to take some one-dimensional probability distribution Q_i for every $i \in N \setminus S$ (for example, take $\mathbf{X}_i = \{0,1\}$, $Q_i(0) = Q_i(1) = 1/2$) and put

$$P = Q \times \prod_{i \in N \setminus S} Q_i.$$

It makes no problem to see that for $\langle E,F|G\rangle \in \mathcal{T}(N)$ the following *simplifying principle* holds (use Definition 2): $E \perp\!\!\!\perp F \,|\, G\,(P)$ iff $(E \cap S) \perp\!\!\!\perp (F \cap S) \,|\, (G \cap S)\,(Q)$, with the convention $\emptyset \perp\!\!\!\perp \bullet \,|\, \bullet\,(Q)$ and $\bullet \perp\!\!\!\perp \emptyset \,|\, \bullet\,(Q)$, where the bullet stands for an arbitrary set of variables here. This key property then easily allows us to evidence the desired facts $A \perp\!\!\!\perp B \,|\, C\,(P)$, $I \perp\!\!\!\perp J \,|\, K\,(P)$ and $\neg[X \perp\!\!\!\perp Y \,|\, Z\,(P)]$.

Let us show in details how the condition (A.1) can be handled. Supposing $X \cap CD \cap KL \neq \emptyset$ choose $p \in X \cap CD \cap KL$ and $q \in Y$ and use the construction [A] (that is we take $S = \{p,q\}$). As $AB \cap \{p,q\} \subset \{q\}$ one has either $A \cap \{p,q\} = \emptyset$ or $B \cap \{p,q\} = \emptyset$ and therefore necessarily $A \cap \{p,q\} \perp\!\!\!\perp B \cap \{p,q\} \,|\, C \cap \{p,q\}\,(Q)$ and hence $A \perp\!\!\!\perp B \,|\, C\,(P)$ by the simplifying principle. Similarly, $IJ \cap \{p,q\} \subset \{q\}$ gives $I \cap \{p,q\} \perp\!\!\!\perp J \cap \{p,q\} \,|\, K \cap \{p,q\}\,(Q)$ and hence $I \perp\!\!\!\perp J \,|\, K\,(P)$. Finally $\neg[\{p\} \perp\!\!\!\perp \{q\} \,|\, \emptyset\,(Q)]$ says $\neg[X \cap \{p,q\} \perp\!\!\!\perp Y \cap \{p,q\} \,|\, Z \cap \{p,q\}\,(Q)]$ and hence $\neg[X \perp\!\!\!\perp Y \,|\, Z\,(P)]$ by the simplifying principle. The subcase $Y \cap CD \cap KL \neq \emptyset$ is analogous: one can choose $p \in Y \cap CD \cap KL$, $q \in X$ and repeat the procedure above with $X \leftrightarrow Y$.

It is left to the reader to verify that also the other conditions can be handled using the respective basic construction – it is a mechanic application of the method used in the case of (A.1). The hints are below.

(A.2) Choose $p \in X \cap A \cap KL$ and $q \in Y \cap ACD$.

(A.3) Choose $p \in X \cap A \cap I$ and $q \in Y \cap A \cap I$.

(B.1) Choose $p \in X$, $q \in Y$, $r \in Z \cap D \cap L$.

(B.2) The case $X \cap D \cap L \neq \emptyset$ was covered by (A.1), suppose $X \cap D \cap L = \emptyset$ and choose $p \in X \cap D$, $q \in Y$, $r \in XZ \cap L$ (owing to $X \cap D \cap L = \emptyset$ we are sure that $p \neq r$).

(B.3) Choose $p \in X \cap CD$, $q \in Y \cap CD$ and $r \in Z \cap L$.

(B.4) Choose $p \in X \cap ACD$, $q \in Y \cap ACD$ and $r \in Z \cap ACD \cap L$.

(C.1) Choose $p \in X$, $q \in Y$ and $r \in W \cap C \cap K$.

(C.2) The case $X \cap C \cap K \neq \emptyset$ was covered by (A.1), suppose $X \cap C \cap K = \emptyset$ and choose $p \in X \cap C$, $q \in Y$, $r \in XW \cap K$ what ensures $p \neq r$.

(C.3) Choose $p \in X \cap CD$, $q \in Y \cap CD$ and $r \in W \cap K$.

(C.4) Take $p \in X \cap ACD$, $q \in Y \cap ACD$ and $r \in W \cap ACD \cap K$.

(D) The case $Z \cap D \cap L \neq \emptyset$ was already solved by (B.1). Suppose $Z \cap D \cap L = \emptyset$ and put $p \in X$, $q \in Y$, $r \in Z \cap D$, $s \in Z \cap L$ (as $Z \cap D \cap L = \emptyset$ we have $r \neq s$).

(E) For the case $W \cap C \cap K \neq \emptyset$ see (C.1). Otherwise choose $p \in X$, $q \in Y$, $r \in W \cap C$, $s \in W \cap K$ (surely $r \neq s$).

(F) Choose $p \in X$, $q \in Y$, $r \in Z \cap D$ and $s \in W \cap D \cap K$.

(G) Choose $p \in X$, $q \in Y$, $r \in Z \cap C \cap L$ and $s \in W \cap C$.

(H) Choose $p \in X \cap ACD$, $q \in Y \cap IKL$, $r \in Z \cap D \cap I$ and $s \in W \cap A \cap K$.

(I) The most of cases was covered by preceding situations. Owing to (C.1) and (F) one can suppose $W \cap AB \cap K \neq \emptyset$ and owing to the permutation $A \leftrightarrow B$ even $W \cap A \cap K \neq \emptyset$. Then (H) enables us to suppose $X \cap B \neq \emptyset \neq Y \cap B$, and (A.2), where B stands for A, leads to $X \cap B \cap IJ \neq \emptyset$ and $\emptyset \neq Y \cap B \cap IJ$. Owing to (A.3) and the permutation $I \leftrightarrow J$ it collapses to the situation $X \cap B \cap I \neq \emptyset \neq Y \cap B \cap J$. Then we choose $p \in X \cap B \cap I$, $q \in Y \cap B \cap J$, $r \in Z \cap D \cap I$, $s \in Z \cap D \cap J$ and $t \in W \cap A \cap K$.

(J) Analogously to the preceding case (B.1), (G) and $I \leftrightarrow J$ allows us to suppose $Z \cap D \cap I \neq \emptyset$. Then (H) leads to $X \cap J \neq \emptyset \neq Y \cap J$, and (A.2), (A.3) with $A \leftrightarrow B$ to $X \cap A \cap J \neq \emptyset \neq Y \cap B \cap J$. Then choose $p \in X \cap A \cap J$, $q \in Y \cap B \cap J$, $r \in Z \cap D \cap I$, $s \in W \cap A \cap K$ and $t \in W \cap B \cap K$.

□

6. Main results

Now, we can summarize the lemmas to obtain the desired claims.

Corollary 13. Supposing $\langle A,B|C\rangle$, $\langle I,J|K\rangle$, $\langle X,Y|Z\rangle \in \mathcal{T}(N)$ the following three conditions are equivalent:

(a) $\{\langle A,B|C\rangle, \langle I,J|K\rangle\} \not\models \langle X,Y|Z\rangle$,
(b) $\{\langle A,B|C\rangle, \langle I,J|K\rangle\} \not\vdash_{sem} \langle X,Y|Z\rangle$,
(c) an isomorphic situation to one of 18 conditions below Lemma 10 occurs.

Proof. Lemma 12 claims (c) ⇒ (a) and (a) ⇒ (b) follows from Lemma 6. Supposing (b) we know by Lemma 10 that one of conditions (i)–(iii) mentioned there occurs. However the condition (i) and (ii) are excluded owing to Lemma 9 and therefore necessarily (iii) holds, what concludes the proof of (b) ⇒ (c). □

Corollary 14. Whenever $u,v,w \in \mathcal{T}(N)$ then $\{u,v\} \models w$ iff $\{u,v\} \vdash_{sem} w$.

Moreover, every probabilistically sound inference rule with at most two antecedents is a consequence of the semigraphoid inference rules.

Proof. Suppose that $\mathcal{R}$ is such an inference rule (for example, with two antecedents), let $(u_1,u_2,u_3) \in \mathcal{R}$ be its instance. By probabilistic soundness of $\mathcal{R}$ we have $\{u_1,u_2\} \models \{u_3\}$. Hence $\{u_1,u_2\} \vdash_{sem} u_3$, i.e., u_3 is derivable from $\{u_1,u_2\}$ by the semigraphoid inference rules. As every instance of $\mathcal{R}$ is already covered by the semigraphoid derivability the whole rule $\mathcal{R}$ is unnecessary. □

Further corollary characterizes the semigraphoid closure of a couple of CI-statements.

Corollary 15. Suppose that $\langle A,B|C\rangle$, $\langle I,J|K\rangle$, $\langle X,Y|Z\rangle \in \mathcal{T}(N)$ and $\mathcal{M} \subset \mathcal{T}(N)$ is the semigraphoid closure of $\{\langle A,B|C\rangle, \langle I,J|K\rangle\}$.

If [$C \cap L \neq \emptyset$ or $D \cap K \neq \emptyset$], then $\langle X,Y|Z\rangle \in \mathcal{M}$ iff it is dominated either by $\langle A,B|C\rangle$, $\langle I,J|K\rangle$ or by their symmetric images.

If [$C \cap L = \emptyset = D \cap K$], then $\langle X,Y|Z\rangle \in \mathcal{M}$ iff one of situations isomorphic to the conditions from Lemma 9 occurs.

Especially, $\mathcal{M}$ has at most 38 dominant triplets and the list of dominant triplets can be obtained by reduction of the list of candidates given by Lemma 9 (see Example 1 illustrating the procedure).

Proof. Supposing $\langle X,Y|Z\rangle \in \mathcal{M}$ one of conditions (i)–(iii) mentioned in Lemma 10 occurs. However, the condition (iii) is excluded by Corrollary 13. In case [$C \cap L \neq \emptyset$ or $D \cap K \neq \emptyset$] just (i) occurs and in case [$C \cap L = \emptyset = D \cap K$] just (ii) occurs. The converse implication follows from Lemma 9. Thus, each triplet in $\mathcal{M}$ is dominated by one of 38 potential triplets described in the remark below Lemma 9. Hence, each dominant triplet must appear in that list of candidates. □

Finally, we derive the main result.

Theorem 16. The semigraphoid closure of a couple of elements of $\mathcal{T}(N)$ is a CI-model.

Proof. Supposing $u, v \in \mathcal{T}(N)$ let $\mathcal{M} \subset \mathcal{T}(N)$ be the semigraphoid closure of $\{u, v\}$. For each $t \in \mathcal{T}(N) \setminus \mathcal{M}$ we have $\{u, v\} \not\vdash_{sem} t$ what is equivalent to $\{u, v\} \not\models t$ by Corollary 13. Thus, one can find a CI-model $\mathcal{M}_t$ with $t \notin \mathcal{M}_t \supset \{u, v\}$. By consecutive application of Lemma 3

$$\mathcal{M} = \bigcap_{t \in \mathcal{T}(N) \setminus \mathcal{M}} \mathcal{M}_t$$

is a CI-model. $\square$

7. Appendix (the proof of Lemma 10)

Let us start with a long remark which somehow responds to reviewers' comments. Perhaps the presented rough description of the method can give at least superficial conception to a reader who wishes to skip the proof.

Remark. I admit that the way of presentation of the proof of Lemma 10 is somehow unusual, it looks like a proof 'by a picture'. The treated situations are described by diagrams similar to the pictures from figure 1, but more complex, because of the type of pictures from figure 1 is not sufficiently general to describe precisely all treated situations. I deliberated about the proof for a long time and came to the conclusion that the only way (I know) to present a complete proof, which is moreover easy to follow and check, is to use the diagrams. Well, in fact, only a part of that huge 'decision tree' is presented, since isomorphic parts are omitted.

One of the reviewers mentioned that one would appreciate describing of the method of the construction of the tree (maybe instead of pursuing nine pages of the proof). The reason why I cannot to do so is that I do not know such an universal method of the construction. I actually came to that tree by an analysis case by case. My aim was either to obtain a situation when $\langle X, Y|Z\rangle$ belongs to the semigraphoid closure of $\langle A, B|C\rangle$ and $\langle I, J|K\rangle$ (that appeared to be the situations described by Lemma 9) or a situation when $\langle X, Y|Z\rangle$ is not probabilistically implied by $\langle A, B|C\rangle$ and $\langle I, J|K\rangle$ (that appeared to be the situations in (iii) from Lemma 10). However, during the time of the construction of the 'decision tree' I did not know the present list of conditions (A.1)–(J). So, in any specific obtained situation I had to ask whether I can already construct a probability distribution P such that $A \perp\!\!\!\perp B \mid C\,(P)$, $I \perp\!\!\!\perp J \mid K\,(P)$ but $\neg[X \perp\!\!\!\perp Y \mid Z\,(P)]$, or whether some further branching is needed. Hence, the list of conditions (A.1)–(J) was the result of that laborious analysis, and therefore I do not know any simple method how to construct the needed decision tree.

Although, perhaps such a method can be found provided that one already knows the required leaves of the decision tree. In fact, many such decision trees can be constructed. On the other hand, I do not consider that it would be fair to omit the proof of Lemma 10 completely and to advice the reader to verify himself/herself that the situations mentioned in Lemma 10 exhaust the general case (say by construction of his/her own decision tree having the prescribed leaves). It is still too complicated task, I guess. And I myself do not want to be on the list of authors who claim something without a proof which cannot be checked!

Note that I had a very general intuitive clue for the construction of the tree. Alas, not sufficient to reconstruct the tree! One starts with a general situation when any of set X, Y, Z, W can intersect any of the 16 'boxes' $A \cap I, A \cap J, \ldots, D \cap K$ (see figure 1). Then one starts to 'clarify' four boxes in the right lower corner, that is the area $CD \cap KL$. By clarification of $CD \cap KL$ is understood that one obtains the situation when $CD \cap KL$ is divided into 'black' conditioned area or 'white' uncovered area (see figure 1) and the branches lead to the leaves of the tree. The corresponding branches reveal only the situations $\boxed{1.}$ in figure 1. Then the right upper corner (that is $CD \cap IJ$) and the left lower corner (that is $AB \cap KL$) are clarified in similar manner. Now, the branches reveal also the situations $\boxed{2.}$ and $\boxed{3.}$ in figure 1. Finally, the left upper corner $AB \cap IJ$ is solved. The situation $\boxed{4.}$ from figure 1 can be obtained only in that final phase.

Proof. To make the formally complicated proof easy to understand and follow a schematic description of the considerations is used. Each specific situation, that is a collection of relationships among triplets $\langle A, B|C\rangle, \langle I, J|K\rangle$ and $\langle X, Y|Z\rangle$, is expressed shortly by means of a table with four rows and columns. The columns correspond to sets A, B, C, D (from left to right in this order) while the rows represent sets I, J, K, L (from top to bottom in this order). The boxes of the table represent their corresponding intersections: for instance the box in the first column and the second row represents $A \cap J$. The letters in a box say which of the sets X, Y, Z, W may have nonempty intersection with the set represented by the box (i.e., which sets were not excluded by some previous considerations). A dash in a box signify that the represented set is already considered to be empty. A bold letter in a box suggests that the corresponding set (i.e., one of X, Y, Z, W) surely has nonempty intersection with the set represented by the box.

The whole proof is in fact a huge 'decision tree' which has the general situation as the root. The leaves of that tree, that is the final situations, will be either situations isomorphic to the conditions [1.]–[4.] from Lemma 9 or situations isomorphic to the conditions (A.1)–(J) listed below Lemma 10. Some of the intermediate situations are labeled, usually by framed Roman numerals, possibly supplemented with a letter.

Every branching path out of a situation corresponds to the conditions which together exhaust that situation. They need not be mutually exclusive in general (as it is the case in classic decision trees). The 'real' branches, that is the branches leading to the right, represent the indicated conditions while the main direction (the 'straight' branch) represents the remaining case, that is the case when none of the conditions, represented by 'real' branches is satisfied. Sometimes, the main (= straight) branch is missing. This means that the real branches already exhaust the situation. This fact usually follows either from the assumption $X \neq \emptyset$ or from the assumption $Y \neq \emptyset$ (because of $\langle X, Y|Z\rangle \in \mathcal{T}(N)$ – see section 2). To help the reader to see the respective reason immediately, the respective argument in square brackets is reminded just under the last (real) branch. That's the meaning of the symbol $[X \neq \emptyset]$ or $[Y \neq \emptyset]$. Anyway, if a straight branch is missing then the lowest right branch plays the role of the 'remaining case'.

Final situations will be denoted by the symbol of the corresponding condition supplemented by a sketchy notation of the corresponding permutation. Concretely, we introduce a shortened notation for every elementary transposition as follows:

$\heartsuit$ $\quad A \leftrightarrow B$,
$\diamondsuit$ $\quad I \leftrightarrow J$,
$\triangle$ $\quad [A, B, C, D] \leftrightarrow [I, J, K, L]$,
$\clubsuit$ $\quad X \leftrightarrow Y$.

However, within this proof we will use also the transposition

$\spadesuit$ $\quad [Z, C, K] \leftrightarrow [W, D, L]$,

and thus utilize a wider group of 32 permutations generated by these five elementary transpositions. We left to the reader to verify that all three conditions (i)–(iii) from Lemma 10 are saved by five above mentioned transpositions (for example, the condition (B.1) can be transformed by $\spadesuit$ into (C.1)). As all branchings in the 'decision tree' are based on 'intersections' of above mentioned sets, many of the situations will be simply 'solved' by a reference to an isomorphic case. For instance, the symbol (A.2)$^{\triangle\diamondsuit}$ will denote the situation (A.2), that is $[X \cap A \cap KL \neq \emptyset$ & $Y \cap ACD \neq \emptyset]$, transformed first by $\triangle$ and then by $\diamondsuit$, that is the situation $[X \cap J \cap CD \neq \emptyset$ & $Y \cap JKL \neq \emptyset]$. The order of transpositions is important, since they do not commute in general.

A	B	C	D	
XYZW	*XYZW*	*XYZW*	*XYZW*	*I*
XYZW	*XYZW*	*XYZW*	*XYZW*	*J*
XYZW	*XYZW*	*XYZW*	*XYZW*	*K*
XYZW	*XYZW*	*XYZW*	*XYZW*	*L*

→ $XY \cap CD \cap KL \neq \emptyset$ → (A.1)
→ $Z \cap D \cap L \neq \emptyset$ → (B.1)
→ $W \cap C \cap K \neq \emptyset$ → (C.1)
↓

I.

A	B	C	D	
XYZW	*XYZW*	*XYZW*	*XYZW*	*I*
XYZW	*XYZW*	*XYZW*	*XYZW*	*J*
XYZW	*XYZW*	*Z*	*ZW*	*K*
XYZW	*XYZW*	*ZW*	*W*	*L*

I.

A	B	C	D	
XYZW	XYZW	XYZW	XYZW	I
XYZW	XYZW	XYZW	XYZW	J
XYZW	XYZW	Z	ZW	K
XYZW	XYZW	ZW	W	L

symmetric with respect to △ and ♠

→ I.a

$Z \cap C \cap L \neq \emptyset$

→ $Z \cap D \neq \emptyset$ → (D)

→ $W \cap C \neq \emptyset$ → (G)

↓

A	B	C	D	
XYZW	XYZW	XYZ	XYW	I
XYZW	XYZW	XYZ	XYW	J
XYZW	XYZW	Z	W	K
XYZW	XYZW	**Z**	W	L

symmetric with respect to ♡

→ I.b

$X \cap ACD \neq \emptyset$

→ $Y \cap ACD \neq \emptyset$ → (B.4)

[$Y \neq \emptyset$] → $Y \cap B \neq \emptyset$ ($Y \cap ACD = \emptyset$)

→ $X \cap BCD \neq \emptyset$ → (B.4)[♡]

↓

A	B	C	D	
XZW	YZW	Z	W	I
XZW	YZW	Z	W	J
XZW	YZW	Z	W	K
XZW	YZW	**Z**	W	L

→ [1.]

i.e. $\langle X, Y|Z\rangle \prec \langle A, B|C\rangle$

[$X \neq \emptyset$] → $X \cap BCD \neq \emptyset$
isomorphic to I.b by ♡

→ $Z \cap D \cap K \neq \emptyset$
isomorphic to I.a
by △

→ $W \cap D \cap K \neq \emptyset$
isomorphic to I.a
by ♠

→ $W \cap C \cap L \neq \emptyset$
isomorphic to I.a
by ♠△

↓

II.

A	B	C	D	
XYZW	XYZW	XYZW	XYZW	I
XYZW	XYZW	XYZW	XYZW	J
XYZW	XYZW	Z	—	K
XYZW	XYZW	—	W	L

II.

A	B	C	D	
XYZW	XYZW	XYZW	XYZW	I
XYZW	XYZW	XYZW	XYZW	J
XYZW	XYZW	Z	—	K
XYZW	XYZW	—	W	L

symmetric with respect to all 5 mentioned transpositions

- → **II.a** $X \cap C \cap I \neq \emptyset$
  - → $Y \cap IKL \neq \emptyset$ → (A.2)$^{\triangle}$
  - → $XW \cap K \neq \emptyset$ → (C.2)
  - [$Y \neq \emptyset$] → $Y \cap J \neq \emptyset$ ($Y \cap IKL = \emptyset = XW \cap K$)
    - → $X \cap CD \cap J \neq \emptyset$ → (A.2)$^{\triangle\diamondsuit}$
    - ↓

A	B	C	D	
XZW	XZW	**X**ZW	XZW	I
XYZW	XYZW	YZW	YZW	J
Z	Z	Z	—	K
XZW	XZW	—	W	L

- → $Y \cap CD \cap J \neq \emptyset$
  - → $X \cap JL \neq \emptyset$ → (A.2)$^{\clubsuit\triangle\diamondsuit}$
  - → $Z \cap L \neq \emptyset$ → (B.3)
  - ↓

A	B	C	D	
XZW	XZW	**X**ZW	XZW	I
YZW	YZW	YZW	YZW	J
Z	Z	Z	—	K
W	W	—	W	L

→ [1.]$^{\triangle}$

i.e. $\langle X, Y|Z\rangle \prec \langle I, J|K\rangle$

- → $Z \cap D \cap J \neq \emptyset$
  - → $X \cap JL \neq \emptyset$ → (B.4)$^{\triangle\diamondsuit}$
  - → $Z \cap L \neq \emptyset$ → (D)
  - → [1.]$^{\triangle}$
- → $W \cap C \cap J \neq \emptyset$
  - → $X \cap JL \neq \emptyset$ → (C.4)$^{\triangle\diamondsuit}$
  - → $Z \cap A \cap L \neq \emptyset$ → (H)$^{\spadesuit\diamondsuit}$
  - → $Z \cap B \cap L \neq \emptyset$ → (H)$^{\spadesuit\diamondsuit\heartsuit}$
  - → [1.]$^{\triangle}$
- ↓

III.

A	B	C	D	
XZW	XZW	**X**ZW	XZW	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
XZW	XZW	—	W	L

- → $Y \cap C \cap I \neq \emptyset$ $\clubsuit$
- → $X \cap C \cap J \neq \emptyset$ $\diamondsuit$
- → $Y \cap C \cap J \neq \emptyset$ $\diamondsuit\clubsuit$
- → $X \cap A \cap K \neq \emptyset$ $\triangle$
- → $Y \cap A \cap K \neq \emptyset$ $\triangle\clubsuit$
- → $X \cap B \cap K \neq \emptyset$ $\triangle\heartsuit$
- → $Y \cap B \cap K \neq \emptyset$ $\triangle\heartsuit\clubsuit$
- → $X \cap D \cap I \neq \emptyset$ $\spadesuit$
- → $Y \cap D \cap I \neq \emptyset$ $\spadesuit\clubsuit$
- → $X \cap D \cap J \neq \emptyset$ $\spadesuit\diamondsuit$
- → $Y \cap D \cap J \neq \emptyset$ $\spadesuit\diamondsuit\clubsuit$
- → $X \cap A \cap L \neq \emptyset$ $\spadesuit\triangle$
- → $Y \cap A \cap L \neq \emptyset$ $\spadesuit\triangle\clubsuit$
- → $X \cap B \cap L \neq \emptyset$ $\spadesuit\triangle\heartsuit$
- → $Y \cap B \cap L \neq \emptyset$ $\spadesuit\triangle\heartsuit\clubsuit$

all above mentioned cases are isomorphic to **II.a** (as indicated)

↓

V.

A	B	C	D	
XYZW	XYZW	ZW	ZW	I
XYZW	XYZW	ZW	ZW	J
ZW	ZW	Z	—	K
ZW	ZW	—	W	L

III.

A	B	C	D	
XZW	XZW	**X**ZW	XZW	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
XZW	XZW	—	W	L

symmetric with respect to ♡
$X \cap C \cap I \neq \emptyset$

- → **III.a**
  $X \cap A \cap L \neq \emptyset$
  - → $Y \cap ACD \neq \emptyset$ → (A.2)
  - → $XZ \cap D \neq \emptyset$ → (B.2)$^{\triangle}$
  - [$Y \neq \emptyset$] → $Y \cap B \cap J \neq \emptyset$ ($XZ \cap D = \emptyset = Y \cap ACD$)
    ↓
    III.b

A	B	C	D	
XZW	XZW	**X**ZW	W	I
XZW	X**Y**ZW	Z	W	J
Z	Z	Z	—	K
XZW	XZW	—	W	L

    $X \cap C \cap I \neq \emptyset \neq Y \cap B \cap J$
    - → $X \cap B \cap L \neq \emptyset$ → (A.2)$^{\heartsuit}$
    - → $X \cap B \cap J \neq \emptyset$ → (A.3)$^{\heartsuit\diamondsuit}$
    - → $Z \cap B \cap L \neq \emptyset$ → (B.4)$^{\heartsuit}$
    - ↓

A	B	C	D	
XZW	XZW	**X**ZW	W	I
XZW	**Y**ZW	Z	W	J
Z	Z	Z	—	K
XZW	W	—	W	L

    → [2.]$^{\triangle\heartsuit\diamondsuit\clubsuit}$
    i.e. $\langle X, Y|Z\rangle \prec \langle (A \setminus K) \cup (I \setminus D), B \cap J | K \cup (C \cap J)\rangle$
- → $X \cap B \cap L \neq \emptyset$ isomorphic to **III.a** by ♡
- → **III.c**
  $Z \cap A \cap L \neq \emptyset$
  - → $X \cap D \neq \emptyset$ → (B.2)
  - → $Y \cap ACD \neq \emptyset$ → (B.4)
  - → $Z \cap D \neq \emptyset$ → (D)
  - [$Y \neq \emptyset$] → $Y \cap B \cap J \neq \emptyset$ ($XZ \cap D = \emptyset = Y \cap ACD$)
    ↓
    III.b
- → $Z \cap B \cap L \neq \emptyset$ isomorphic to **III.c** by ♡
- ↓

IV.

A	B	C	D	
XZW	XZW	XZW	XZW	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

IV.

A	B	C	D	
XZW	XZW	XZW	XZW	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

symmetric with respect to ♡

→ $Y \cap A \cap J \neq \emptyset \neq Y \cap B \cap J$

→ $X \cap A \cap J \neq \emptyset$ → (A.3)$^{\diamondsuit}$

→ $X \cap B \cap J \neq \emptyset$ → (A.3)$^{\diamondsuit\heartsuit}$

A	B	C	D	
XZW	XZW	XZW	XZW	I
YZW	**Y**ZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

→ [1.]$^{\triangle}$

i.e. $\langle X, Y|Z\rangle \prec \langle I, J|K\rangle$

→ IV.a

$Y \cap A \cap J \neq \emptyset$ $(Y \cap B \cap J = \emptyset)$

→ $X \cap A \cap J \neq \emptyset$ → (A.3)$^{\diamondsuit}$

A	B	C	D	
XZW	XZW	XZW	XZW	I
YZW	XZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

→ [3.]$^{\triangle\diamondsuit\clubsuit}$

i.e. $\langle X, Y|Z\rangle \prec \langle I \cup (B \cap J), A \cap J | K \cup (C \cap J)\rangle$

→ $Y \cap B \cap J \neq \emptyset$ $(Y \cap A \cap J = \emptyset)$ isomorphic to IV.a by ♡

$[Y \neq \emptyset]$

V.

A	B	C	D	
XYZW	XYZW	ZW	ZW	I
XYZW	XYZW	ZW	ZW	J
ZW	ZW	Z	—	K
ZW	ZW	—	W	L

symmetric with respect to ♡, ♢, △, ♠

→ V.a

$Z \cap D \cap I \neq \emptyset$

→ $Z \cap L \neq \emptyset$ → (D)

→ VI.

A	B	C	D	
XYZW	XYZW	ZW	**Z**W	I
XYZW	XYZW	ZW	ZW	J
ZW	ZW	Z	—	K
W	W	—	W	L

→ $Z \cap D \cap J \neq \emptyset$ ♢

→ $Z \cap A \cap L \neq \emptyset$ △

→ $Z \cap B \cap L \neq \emptyset$ △♡

→ $W \cap C \cap I \neq \emptyset$ ♠

→ $W \cap C \cap J \neq \emptyset$ ♠♢

→ $W \cap A \cap K \neq \emptyset$ ♠△

→ $W \cap B \cap K \neq \emptyset$ ♠△♡

all above mentioned cases are isomorphic to V.a (as indicated)

X.

A	B	C	D	
XYZW	XYZW	Z	W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

VI.

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*ZW*	*ZW*	*J*
ZW	*ZW*	*Z*	—	*K*
W	*W*	—	*W*	*L*

$Z \cap D \cap I \neq \emptyset$

→ $W \cap C \cap J \neq \emptyset$

→ $W \cap K \neq \emptyset$ → (E)

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*Z**W***	*ZW*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

→ $X \cap I \neq \emptyset$

→ $Y \cap I \neq \emptyset$ → (B.4)$^{\triangle}$

[$Y \neq \emptyset$] → $Y \cap J \neq \emptyset$ ($Y \cap I = \emptyset$)

→ $X \cap J \neq \emptyset$ → (C.4)$^{\triangle\diamond}$

A	B	C	D	
XZW	*XZW*	*ZW*	***Z**W*	*I*
YZW	*YZW*	*Z**W***	*ZW*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

i.e. $\langle X, Y|Z\rangle \prec \langle I, J|K\rangle$ → [1.]$^{\triangle}$

[$X \neq \emptyset$] → $X \cap J \neq \emptyset$ ($X \cap I = \emptyset$)

→ $Y \cap J \neq \emptyset$ → (C.4)$^{\triangle\diamond}$

A	B	C	D	
YZW	*YZW*	*ZW*	***Z**W*	*I*
XZW	*XZW*	*Z**W***	*ZW*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

i.e. $\langle X, Y|Z\rangle \prec \langle J, I|K\rangle$ → [1.]$^{\triangle\diamond}$

VII.

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*Z*	*ZW*	*J*
ZW	*ZW*	*Z*	—	*K*
W	*W*	—	*W*	*L*

VII.

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*Z*	*ZW*	*J*
ZW	*ZW*	*Z*	—	*K*
W	*W*	—	*W*	*L*

$Z \cap D \cap I \neq \emptyset$

→ $Z \cap D \cap J \neq \emptyset$

→ → $W \cap K \neq \emptyset$ ⟶ (I)

↓

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*Z*	***Z**W*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

→ $X \cap I \neq \emptyset$

→ → $Y \cap I \neq \emptyset$ ⟶ (B.4)$^{\triangle}$

[$Y \neq \emptyset$] → $Y \cap J \neq \emptyset$ ($Y \cap I = \emptyset$)

→ → $X \cap J \neq \emptyset$ ⟶ (B.4)$^{\triangle\diamond}$

↓

A	B	C	D	
XZW	*XZW*	*ZW*	***Z**W*	*I*
YZW	*YZW*	*Z*	***Z**W*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

⟶ [1.]$^{\triangle}$

i.e. $\langle X, Y|Z\rangle \prec \langle I, J|K\rangle$

[$X \neq \emptyset$] → $X \cap J \neq \emptyset$ ($X \cap I = \emptyset$)

→ → $Y \cap J \neq \emptyset$ ⟶ (B.4)$^{\triangle\diamond}$

↓

A	B	C	D	
YZW	*YZW*	*ZW*	***Z**W*	*I*
XZW	*XZW*	*Z*	***Z**W*	*J*
Z	*Z*	*Z*	—	*K*
W	*W*	—	*W*	*L*

⟶ [1.]$^{\triangle\diamond}$

i.e. $\langle X, Y|Z\rangle \prec \langle J, I|K\rangle$

↓

VIII.

A	B	C	D	
XYZW	*XYZW*	*ZW*	***Z**W*	*I*
XYZW	*XYZW*	*Z*	*W*	*J*
ZW	*ZW*	*Z*	—	*K*
W	*W*	—	*W*	*L*

VIII.

	A	B	C	D	
	XYZW	XYZW	ZW	**Z**W	I
	XYZW	XYZW	Z	W	J
	ZW	ZW	Z	—	K
	W	W	—	W	L

symmetric with respect to ♡
$Z \cap D \cap I \neq \emptyset$

- → VIII.a
  $W \cap A \cap K \neq \emptyset$
  - → $W \cap C \neq \emptyset$ → (E)
  - → $W \cap B \cap K \neq \emptyset$ → (J)
  - ↓

	A	B	C	D	
	XYZW	XYZW	Z	**Z**W	I
	XYZW	XYZW	Z	W	J
	Z**W**	Z	Z	—	K
	W	W	—	W	L

  - → $X \cap I \neq \emptyset$
    - → $Y \cap I \neq \emptyset$ → (B.4)[△]
    - → $Y \cap A \cap J \neq \emptyset$ → (H)[♣]
    - [$Y \neq \emptyset$] → $Y \cap B \cap J \neq \emptyset$ $(Y \cap I = \emptyset = Y \cap A \cap J)$
      - → $X \cap B \cap J \neq \emptyset$ → (A.3)[◇♡]
      - ↓

	A	B	C	D		
	XZW	XZW	Z	**Z**W	I	
	XZW	**Y**ZW	Z	W	J	→ [2.][◇♡♣]
	Z**W**	Z	Z	—	K	
	W	W	—	W	L	

i.e. $\langle X, Y|Z\rangle \prec \langle (I \setminus C) \cup (A \setminus L), B \cap J | C \cup (B \cap K)\rangle$

  - → $X \cap A \cap J \neq \emptyset$
    - → $Y \cap A \cap J \neq \emptyset$ → (A.3)[◇]
    - → $Y \cap I \neq \emptyset$ → (H)
    - [$Y \neq \emptyset$] → $Y \cap B \cap J \neq \emptyset$ $(Y \cap I = \emptyset = Y \cap A \cap J)$
      - → $X \cap B \cap J \neq \emptyset$ → (A.3)[◇♡]
      - → [2.][◇♡♣]
  - [$X \neq \emptyset$] → $X \cap B \cap J \neq \emptyset$ $(X \cap I = \emptyset = X \cap A \cap J)$
    - → $Y \cap B \cap J \neq \emptyset$ → (A.3)[◇♡]
    - ↓

	A	B	C	D		
	YZW	YZW	Z	**Z**W	I	
	YZW	**X**ZW	Z	W	J	→ [2.][◇♡]
	Z**W**	Z	Z	—	K	
	W	W	—	W	L	

i.e. $\langle X, Y|Z\rangle \prec \langle B \cap J, (I \setminus C) \cup (A \setminus L) | C \cup (B \cap K)\rangle$

- → $W \cap B \cap K \neq \emptyset$
  isomorphic to VIII.a
  by ♡
- ↓

IX.

	A	B	C	D	
	XYZW	XYZW	ZW	**Z**W	I
	XYZW	XYZW	Z	W	J
	Z	Z	Z	—	K
	W	W	—	W	L

IX.

A	B	C	D	
XYZW	XYZW	ZW	**Z**W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

symmetric with respect to ♡
$Z \cap D \cap I \neq \emptyset$

→ $X \cap I \neq \emptyset$

→ $Y \cap I \neq \emptyset$ → (B.4)[△]

IX.a

A	B	C	D	
XZW	XZW	ZW	**Z**W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

→ IV.

→ $X \cap J \neq \emptyset$ $(X \cap I = \emptyset)$ isomorphic to IX.a by ♣

$[X \neq \emptyset]$

X.

A	B	C	D	
XYZW	XYZW	Z	W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

symmetric with respect to ♡ and ♢

→ X.a

$X \cap A \cap I \neq \emptyset$

→ $Y \cap A \cap I \neq \emptyset$ → (A.3)

A	B	C	D	
XZW	XYZW	Z	W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

symmetric with respect to △

→ X.b

$X \cap B \cap I \neq \emptyset$

→ $Y \cap B \cap I \neq \emptyset$ → (A.3)[♡]

A	B	C	D	
XZW	**X**ZW	Z	W	I
XYZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

→ IV.

→ $X \cap A \cap J \neq \emptyset$ isomorphic to X.b by △

XI.

A	B	C	D	
XZW	YZW	Z	W	I
YZW	XYZW	Z	W	J
Z	Z	Z	—	K
W	W	—	W	L

→ $X \cap B \cap I \neq \emptyset$ ♡

→ $X \cap A \cap J \neq \emptyset$ ♢

→ $X \cap B \cap J \neq \emptyset$ ♡♢

$[X \neq \emptyset]$

all above mentioned cases are isomorphic to X.a (as indicated)

XI.

	A	B	C	D
I	**X**ZW	YZW	Z	W
J	YZW	XYZW	Z	W
K	Z	Z	Z	—
L	W	W	—	W

→ $Y \cap B \cap J \neq \emptyset$

→ $X \cap B \cap J \neq \emptyset$ ⟶ (A.3)$^{\heartsuit\diamondsuit}$

	A	B	C	D
I	**X**ZW	YZW	Z	W
J	YZW	**Y**ZW	Z	W
K	Z	Z	Z	—
L	W	W	—	W

⟶ [2.]

i.e. $\langle X, Y|Z\rangle \prec \langle A \cap I, (J \setminus C) \cup (B \setminus L)|C \cup (A \cap K)\rangle$

	A	B	C	D
I	**X**ZW	YZW	Z	W
J	YZW	XZW	Z	W
K	Z	Z	Z	—
L	W	W	—	W

⟶ [4.]

i.e. $\langle X, Y|Z\rangle \prec \langle (A \cap I) \cup (B \cap J), (A \cap J) \cup (B \cap I)|C \cup K\rangle$

□

8. Conclusions

As the concept of CI belongs to fundamentals of probability theory, the related question of axiomatic characterization of formal properties of CI, treated in this paper, can be of interest to the researchers in probability theory. The presented result about relative completeness of the semigraphoid inference rules (Corollary 14) has theoretical significance for one of basic calculi for uncertainty handling in artificial intelligence – for probabilistic reasoning. It confirms Pearl's and Paz's revised conjecture that any probabilistically sound inference rule with at most two antecedents is a consequence of the semigraphoid properties. One can interpret this result by saying that semigraphoids are "two-antecedental" approximations of CI-models [24]. Note for information that the only independent (i.e., not covered by the semigraphoid derivability) probabilistically sound inference rule with 3 antecedents known so far is the following one (see [21]):

$$[\langle A, B|C\rangle \ \&\ \langle A, C|D\rangle \ \&\ \langle A, D|B\rangle] \longrightarrow \langle A, C|B\rangle.$$

Nevertheless, our main result may have also more concrete significance. One can sometimes face the task to show that a given independency model is a CI-model. This problem can appear both in probabilistic reasoning and in multivariate statistics, where analogous structural models are used. The presented theorem gives a relatively simple sufficient condition. Thus, instead of laborious construction of a probability distribution it may be sufficient to check whether the given independency model is the semigraphoid closure of a proper couple of triplets. Its characterization given in Corollary 15 is then beneficial. To test by a computer whether a given triplet describing a CI-statement belongs to the semigraphoid closure of a considered couple of triplets one can first generate the list of dominant triplets of the semigraphoid closure of the couple and then to check whether the given triplet is dominated by one of those dominant triplets. In general, nice simple record of any semigraphoid (not only of the semigraphoid closure of a couple of triplets) may be the list of its dominant triplets.

Acknowledgements

I am grateful to both reviewers of the paper for their useful comments which helped me to improve the presentation. Similar thanks are devoted to my colleague F. Matúš for a few terse remarks and to that former *Artificial Intelligence Journal* reviewer who showed positive attitude and gave some valuable comments on a previous version of this paper submitted originally to that journal.

References

[1] J.N. Darroch, S.L. Lauritzen and T.P. Speed, Markov field theory and log-linear interaction models for contingency tables, Ann. Statist. 8 (1980) 522–539.

[2] A.P. Dawid, Conditional independence in statistical theory, J. Roy. Statist. Soc., Series B 41 (1979) 1–31.

[3] L.C. van der Gaag and J.-J. Ch. Meyer, Characterising normal forms for informational independence, in: *Proc. of IPMU'96, Granada*, July 1–July 5, 1996, Vol. II, pp. 973–978.

[4] D. Geiger, A. Paz and J. Pearl, Axioms and algorithms for inferences involving probabilistic independence, Informat. and Comput. 1 (1991) 128–141.

[5] D. Geiger and J. Pearl, Logical and algorithmic properties of conditional independence and their application to Bayesian networks, Ann. Math. and AI 2 (1990) 165–178.

[6] D. Geiger and J. Pearl, Logical and algorithmic properties of conditional independence and graphical models, Ann. Statist. 21 (1993) 2001–2021.

[7] H. Kiiveri, T.P. Speed and J.B. Carlin, Recursive causal models, J. Austral. Math. Soc., Series A 36 (1984) 30–51.

[8] S.L. Lauritzen and N. Wermuth, Graphical models with associations between variables, some of which are quantitative and some qualitative, Ann. Statist. 17 (1989) 31–57.

[9] F.M. Malvestuto, A unique formal system for binary decomposition of database relations, probability distributions and graphs, Inform. Sci. 59 (1992) 21–52. Also: F.M. Malvestuto and M. Studený, Comment on "A unique formal . . . graphs", Inform. Sci. 63 (1992) 1–2.

[10] F. Matúš, Probabilistic conditional independence structures and matroid theory: backgrounds, Internat, J. Gen. Systems 22 (1994) 185–196.

[11] F. Matúš, Stochastic independence, algebraic independence and abstract connectedness, Theoret. Comput. Sci. 134 (1994) 455–471.

[12] J. Pearl, *Probabilistic Reasoning in Intelligent Systems* (Morgan-Kaufmann, 1988).

[13] J. Pearl and A. Paz, Graphoids: a graph-based logic for reasoning about relevance relations, in: *Advances in Artificial Intelligence II*, eds. B. Du Boulay, D. Hogg and L. Steels (North-Holland, 1987) pp. 357–363.

[14] A. Perez and R. Jiroušek, Construction an intensional expert system INES, in: *Medical Decision Making: Diagnostic Strategies and Expert systems*, eds. J.H. von Bemmel, F. Grémy and J. Zvárová (North-Holland, 1985) pp. 307–315.

[15] Y. Sagiv and S.F. Walecka, Subset dependencies and completeness result for a subclass of embedded multivalued dependencies, J. Assoc. Comp. Machinery 29 (1982) 103–117.

[16] R.D. Shachter, Evaluating influence diagrams, Oper. Res. 34 (1986) 871–882.

[17] P.P. Shenoy, Conditional independence in valuation-based systems, Internat. J. Approx. Reason. 10 (1994) 203–234.

[18] J.Q. Smith, Influence diagrams for statistical modelling, Ann. Statist. 17 (1989) 654–672.

[19] W. Spohn, Stochastic independence, causal independence and shieldability, J. Philos. Logic 9 (1980) 73–99.

[20] M. Studený, Multiinformation and the problem of characterization of conditional independence relations, Problems Control Inform. Theory 18 (1989) 3–16.

[21] M. Studený, Conditional independence relations have no finite complete characterization, in: *Information Theory, Statistical Decision Functions and Random Processes, Transaction of the Eleventh Prague Conference B*, eds. S. Kubík and J.Á. Víšek (Kluwer, 1992) or (Academia, 1992) pp. 377–396.

[22] M. Studený, Formal properties of conditional independence in different calculi of AI, in: *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, eds. M. Clarke, R. Kruse and S. Moral (Springer-Verlag, 1993) pp. 341–348.

[23] M. Studený, Description of structures of stochastic conditional independence by means of faces and imsets (a series of 3 papers), 1st Part: Introduction and basic concepts, 2nd Part: Basic theory, 3rd Part: Examples of use and appendices, Internat. J. Gen. Systems 23 (1994/1995) 123–137, 201–219, 323–341.

[24] M. Studený, Semigraphoids are two-antecedental approximations of stochastic conditional independence models, in: *Uncertainty in Artificial Intelligence, Proceedings of the Tenth Conference*, eds. R.L. de Mantaras and D. Poole (Morgan-Kaufmann, 1994) pp. 546–552.

[25] J. Whittaker, *Graphical Models in Applied Multivariate Statistics* (John Wiley, 1990).