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CONVEX CONES IN FINITE–DIMENSIONAL REAL VECTOR SPACES

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Various classes of finite-dimensional closed convex cones are studied. Equivalent characterizations of pointed cones, pyramids and rational pyramids are given. Special class of regular cones, corresponding to "continuous linear" quasiorderings of integer vectors is introduced and equivalently characterized. It comprehends both pointed cones and rational pyramids. Two different ways of determining of vector quasiorderings are dealt with: establishing (i. e. prescribing a set of 'positive' vectors) and inducing through scalar product. The existence of the least finite set of normalized integer vectors establishing every finitely establishable (or equivalently finitely inducable) ordering of integer vectors is shown. For every quasiordering of integer vectors established by a finite exhaustive set there exists the least finite set of normalized integer vectors inducing it and elements of this set can be distinguished by corresponding 'positive' integer vectors.

1. INTRODUCTION

Various classes of closed convex cones in finite-dimensional real vector spaces form the topic of this paper. The source of motivation for this study is in apparently remote area of mathematics, namely in artificial intelligence. Within the framework of our research project[1] we endeavour to develop a convenient mathematical theory to describe structures of conditional stochastic independence of finite number of random variables (this would be of great importance for probabilistic expert systems, a growingly-popular area of artificial intelligence). Nevertheless, the systematic thorough buildup of this theory made in [8] requires some subsidiary results concerning the above mentioned cones (more concretely, several results concerning "continuous linear" quasiorderings on the set of integer vectors are needed and these results stem from other results about convex cones of real vectors). Though these properties of geometric nature look natural, precise proofs require adequate space. As they are rather specific in the theory on conditional independence structures they would complicate the main text.

On the other hand, we don't know any publication where the theory of finitely-dimensional convex cones (especially of rational pyramids) is systematically developed up to the degree sufficient for above mentioned purposes (although some particular results probably can be scattered in the literature).

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Thus, the paper is intended as an adequate treatise (maybe rather technical) on closed convex cones in $\mathbb{R}^n$ (= the set of n-tuples of real numbers), i.e. the cones corresponding to continous linear quasiorderings of real vectors. We tried to base the paper on well-known facts from textbooks of linear algebra, topology and linear programming. Provided we knew that some properties were shown there and we haven't really short proofs of them, we preferred to refer to the corresponding source. Then the property is formulated as *Statement* in the text. Some evident or easy properties often used later are named *Facts*.

Every closed cone in $\mathbb{R}^n$ corresponds naturally to a quasiordering of n-dimensional real vectors. Namely, having a closed cone $K \subset \mathbb{R}^n$ and real vectors $u, v \in \mathbb{R}^n$ write $v \preceq_K u$ iff $(u - v) \in K$. This defines a reflexive transitive binary relation on $\mathbb{R}^n$ (i.e. quasiordering), which is moreover linear and continous (for details see [6]). Two ways of determining of quasiorderings of vectors are studied in this article: establishing and inducing.

The method of establishing consists in prescribing a set of vectors and considering the 'minimal' quasiordering making these vectors 'positive'. This leads to the concept of conic hull recalled in § 3. On the other hand, the method of inducing by a given set $L \subset \mathbb{R}^n$ consists in declaring vectors having nonnegative scalar products with elements of L to be 'positive'. This leads to the concept of dual cone treated in § 4. In case that a quasiordering is moreover antisymmetric, it is called ordering. The corresponding cones, called pointed, are studied in § 5. The next section (§ 6) introduces a wider class of regular cones, which are later shown to correspond to quasiorderings of integer vectors. Some facts concerning extreme rays studied in § 7 are utilized in § 8 to show several results about further special class of closed cones, namely pyramids and rational pyramids. Finally, quasiorderings of integer vectors, i.e. vectors whose components are integers, are studied in the last section (§ 9).

More detailed comment of contents starts every section.

2. BASIC NOTATION

The set of real, resp. rational, resp. integer, numbers will be denoted by $\mathbb{R}$, resp. $\mathbb{Q}$, resp. $\mathbb{Z}$, the corresponding subsets of nonnegative numbers (including zero) by $\mathbb{R}^+$, resp. $\mathbb{Q}^+$, resp. $\mathbb{Z}^+$. Similarly, the sets of corresponding n-tuples will be denoted by $\mathbb{R}^n$, resp. $\mathbb{Q}^n$, resp. $\mathbb{Z}^n$. The set of positive integers or natural numbers (i.e. $\{1, 2, \dots\}$) will be denoted by $\mathbb{N}$.

The Euclidean norm of a vector x will be denoted by $\|x\|$, the scalar product of vectors x and y by $\langle x, y\rangle$, their sum by $x + y$; the product of a scalar α and a vector x will be written as $\alpha \cdot x$. The symbol $x_k \to x$ means that the sequence $\{x_k\}$ converges to the element x.

Having a set $A \subset \mathbb{R}^n$ the symbol $\overline{A}$ denotes its closure (with respect to the Euclidean norm), $Lin(A)$ its linear hull, $A^\perp$ its orthogonal complement, $(-A)$ its multiple by (-1), i.e. $(-A) = \{-a\,;\, a \in A\}$. Finally, $A \oplus B$ denotes the direct product of sets A and B. The other symbols will be introduced in the text.

Notice: Throughout the paper only real vector spaces $\mathbb{R}^n$ where $n \geq 1$ will be dealt with.

3. CONIC HULL

Basic concepts of cone, closed cone, conic hull and closed conic hull are recalled in this section. This is supplied by a familiar result that the conic hull of a finite set is closed (Proposition 1).

Definition 1. (cone, closed cone)
A set $K \subset \mathbb{R}^n$ is a *cone* iff it satisfies:

$$u, v \in K \implies u + v \in K \tag{1}$$
$$u \in K,\ \alpha \in \mathbb{R}^+ \implies \alpha \cdot u \in K. \tag{2}$$

If K is moreover closed with respect the Euclidean topology (i. e. given by norm) it is a *closed cone.*

Remark. Some authors [7] use term 'convex cone' for sets satisfying (1), (2), while by 'cone' they understand sets satisfying (2). But we are interested in cones corresponding to linear quasiorderings on $\mathbb{R}^n$ (see [6]).

It is evident that intersection of arbitrary nonempty collection of cones is a cone, too. Similarly for closed cones. As the whole space $\mathbb{R}^n$ is a closed cone, for every $L \subset \mathbb{R}^n$ the collection of (closed) cones containing L is nonempty. Thus, the following definitions are correct.

Definition 2. (conic hull, closed conic hull)
Having $L \subset \mathbb{R}^n$ by $con(L)$ denote the least cone containing L. It will be called the *conic hull of* L. The least closed cone containing L will be denoted by $\overline{con}(L)$ and called the *closed conic hull of* L.

It makes no problem to verify:

Fact 1. $con(\emptyset) = \emptyset$ and having $\emptyset \neq L \subset \mathbb{R}^n$ it holds:
$con(L) = \{v \in \mathbb{R}^n \ ;\ v = \sum_{u \in K} \alpha_u \cdot u \text{ where } \emptyset \neq K \subset L \text{ is finite},\ \ \alpha_u \in \mathbb{R}^+\}$.
Fact 2. Having $L \subset \mathbb{R}^n$ its closed conic hull $\overline{con}(L)$ coincides with the closure of its conic hull i. e. $\overline{con(L)}$.

Hint: Verify that the closure of a cone is a cone.

To prove the mentioned result about conic hull of finite sets the following lemma will be used.

Lemma 1. Let $\emptyset \neq K \subset \mathbb{R}^n$ be a closed cone, $v \in \mathbb{R}^n \setminus (-K)$. Then
$\overline{con}(\{v\}) = \{\alpha \cdot v \ ;\ \alpha \in \mathbb{R}^+\}$ and $\overline{con}(K \cup \{v\}) = \{u + \alpha \cdot v \ ;\ u \in K\ \ \alpha \in \mathbb{R}^+\}$.

Proof. Note that the set $A = \{\alpha \cdot v \ ;\ \alpha \in \mathbb{R}^+\}$ is closed. It suffices to make sure that $B = \{u + \alpha \cdot v \ ;\ u \in K\ \ \alpha \in \mathbb{R}^+\}$ is closed. Clearly $A \subset B$. Let $x_k = u_k + \alpha_k \cdot v \in B$ converges to $x \in \mathbb{R}^n$. Suppose that $u_k \neq 0$ for all indices (in case

$u_k \neq 0$ for finite number of indices $x \in \overline{A} = A$, otherwise consider the corresponding subsequence of $\{x_k\}$). Put $\beta_k = \|u_k\| > 0$, $\gamma_k = \alpha_k \beta_k^{-1} \geq 0$, $\tilde{u}_k = \beta_k^{-1} \cdot u_k$. Evidently $x_k = \beta_k \cdot (\tilde{u}_k + \gamma_k \cdot v)$ and $\tilde{u}_k \in K$ $\|\tilde{u}_k\| = 1$. As $\{u \in K;\ \|u\| = 1\}$ is a compact set there exists a convergent subsequence of $\tilde{u}_k$. Thus, without loss of generality, suppose $\tilde{u}_k \to u \in K$, $\|u\| = 1$. In case $\limsup_{k\to\infty} \beta_k = \infty$ (consider directly $\beta_k \to \infty$) it holds $\|\tilde{u}_k + \gamma_k \cdot v\| \to 0$ and hence $\gamma_k \cdot v \to -u$, i. e. $(-u) \in \overline{A} = A$. This contradicts the assumption $v \notin (-K)$. Thus $\{\beta_k\}$ is a bounded sequence and has a convergent subsequence; consider it instead of $\{\beta_k\}$. In case $\beta_k \to 0$ it holds $\beta_k \cdot \tilde{u}_k \to 0$ and $x \in \overline{A} = A$. In case $\beta_k \to \beta > 0$ get $\tilde{u}_k + \gamma_k \cdot v \to \beta^{-1} \cdot x$. Hence $\gamma_k \cdot v \to \beta^{-1} \cdot x - u$ gives $\beta^{-1} \cdot x - u \in \overline{A} = A$ i. e. $x = \beta \cdot (u + \gamma \cdot v)$ for some $\gamma \geq 0$. □

Note that the assumption $v \notin (-K)$ in the preceding lemma is essential. It is illustrated by the following example.

Example. Consider $n = 3$ and put
$K_1 = \{\, (x_1, x_2, x_3)\,;\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 \cdot x_3 \geq x_2^2\ \}$
$K_2 = \{\, (0, 0, x_3)\,;\ x_3 \leq 0\ \}$
(the set K_1 is the closed conic hull of the branch $x_1 \cdot x_3 = 1$, $x_1, x_3 > 0$ of the hyperbola lying in the plane $x_2 = 1$). Both these sets are closed cones, but their sum
$K_1 + K_2 = \{\, (x_1, x_2, x_3)\,;\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}\ \} \setminus \{\, (0, x_2, x_3)\,;\ x_2 > 0\ x_3 \in \mathbb{R}\ \}$
is not 'closed'.

Proposition 1. Let $L \subset \mathbb{R}^n$ finite. Then $\overline{con}(L) = con(L)$.

P r o o f. I. The statement holds under the additional assumption that L is linearly independent.
Indeed: In case $L = \emptyset$ it is evident. Proceed by induction according to $card\, L$; if $L \neq \emptyset$ choose $v \in L$, put $K = \overline{con}(L \setminus \{v\}) = con(L \setminus \{v\})$ (use the induction assumption). As L is linearly independent using Lemma 1 get $\overline{con}(L) = con(L)$.

II. $con(L) = \bigcup\{con(T);\ T \subset L\ \ T \text{ is linearly independent}\}$ in case $L \setminus \{0\} \neq \emptyset$.
Indeed: Clearly $0 \in con(\{y\})$ for any $y \in L \setminus \{0\}$; having $0 \neq x \in con(L)$ use Fact 1 and consider a specification $x = \sum\limits_{u \in K} \alpha_u \cdot u$ $(K \subset L)$ with minimal number of strictly positive α_u. As $T = \{u \in K,\ \alpha_u > 0\} \neq 0$ it suffices to show that T is linearly independent. By contradiction, in opposite case write $0 = \sum\limits_{u \in T} \lambda_u \cdot u$ where $\max\limits_{u \in T} \lambda_u > 0$. Putting $\beta = \max\limits_{u \in T} \lambda_u \alpha_u^{-1}$ get $x = \sum\limits_{u \in T} \alpha_u \cdot u - \beta^{-1} \cdot (\sum\limits_{u \in T} \lambda_u \cdot u) = \sum\limits_{u \in T} (\alpha_u - \lambda_u \beta^{-1}) \cdot u$. As $\alpha_u - \lambda_u \beta^{-1} \geq 0$ for all $u \in T$ and at least one of these numbers is zero, this contradicts the assumption that $card\, T$ is the minimal number of strictly positive coefficients in specifications of x.

III. The statement is trivial in cases $L = \emptyset$ or $L = \{0\}$; in case $L \setminus \{0\} \neq \emptyset$ it follows from I and II as the union of finite number of closed sets is closed. □

Remark. The reader probably recognized that the operation of closed conic hull realizes the idea of establishing mentioned in the Introduction: the situation

$K = \overline{con}(E)$ means that the set E establishes the closed cone K and therefore the corresponding quasiordering. The previous assertion says that in case of finite establishing set conic hull gives the same result, i. e. every 'positive' vector can be directly 'combined' from elements of the establishing set.

4. DUAL CONE

Every subset of $\mathbb{R}^n$ can induce a nonempty closed cone through scalar product as mentioned in the Introduction. The ascribed cone is called dual. The section is devoted to simple properties of this basic procedure of forming cones.

Definition 3. (dual cone)
Let $L \subset \mathbb{R}^n$. Introduce its *dual cone* L^* as follows:
$L^* = \{x \in \mathbb{R}^n;\ \forall\, u \in L\ \ \langle x, u\rangle \geq 0\}$.

Fact 3. Whenever $L \subset \mathbb{R}^n$ then L^* is a nonempty closed cone.

Hint: In case $L \neq \emptyset$ write $L^* = \bigcap_{u \in L} \{x \in \mathbb{R}^n;\ \ \langle x, u\rangle \geq 0\}$ and each of these sets is a closed cone containing 0.

Statement 1. Let L be a nonempty closed cone, $a \in \mathbb{R}^n \setminus L$. Then there exists $p \in L^*$ such that $\langle p, a\rangle < 0$.

Comment: This is a familiar consequence of the Hahn-Banach theorem (see [6] § 14) known as a conic version of well-known separation hyperplane theorem. The reader can find it in this form in [7] as Consequence 11.7.1 or use Theorem 4.5 in [2] resp. Theorem 2.3 in [1].

Some useful facts concerning dual cones follow.

Fact 4. Whenever $L_1 \subset L_2 \subset \mathbb{R}^n$, then $L_1^* \supset L_2^*$ and hence $L_1^{**} \subset L_2^{**}$.

Fact 5. Having $L \subset \mathbb{R}^n$ it holds $L \subset L^{**}$.

Consequence 1. Having $K \subset \mathbb{R}^n$ the following three conditions are equivalent:
(i) K is a nonempty closed cone
(ii) $K = K^{**}$
(iii) $K = L^*$ for some $L \subset \mathbb{R}^n$.

Proof. (i) $\Longrightarrow$ (ii) By Fact 5 $K \subset K^{**}$. Conversely, having $a \in \mathbb{R}^n \setminus K$ by Statement 1 find $p \in K^*$ with $\langle a, p\rangle = \langle p, a\rangle < 0$ i. e. $a \notin K^{**}$. Together $K = K^{**}$. (ii) $\Longrightarrow$ (iii) is evident, (iii) $\Longrightarrow$ (i) follows from Fact 3. □

Fact 6. Whenever $\emptyset \neq L \subset \mathbb{R}^n$ it holds $L^{**} = \overline{con}(L)$.

Hint: $\overline{con}(L) \subset L^{**}$ using Fact 5 and Fact 3. Conversely, having a closed cone K containing L, Fact 4 and Consequence 1 give $L^{**} \subset K$.

Fact 7. Whenever $L \subset \mathbb{R}^n$ then $L^* = L^{***}$.

Hint: Fact 3 and Consequence 1.

Fact 8. Whenever $L \subset \mathbb{R}^n$ then $L^* = con(L)^* = \overline{con}(L)^*$.

Hint: Write $L \subset con(L) \subset \overline{con}(L)$, apply Fact 4; in case $L \neq \emptyset$ Facts 7,6 give $L^* = L^{***} = \overline{con}(L)^*$.

5. POINTED CONE

The antisymmetry condition means that the only simultaneously 'positive' and 'negative' vector is zero vector. The corresponding cones, called pointed cones, are studied in this section. Firstly, the corresponding version of separation hyperplane theorem (Consequence 2) is derived. Then it is used to derive equivalent characterizations of pointed cones (Proposition 2) saying that pointed cones are 'strictly contained' in a halfspace.

Definition 4. (pointed cone)
A nonempty closed cone $K \subset \mathbb{R}^n$ is called *pointed* iff $K \cap (-K) = \{0\}$, i. e.
$[\, u \in K$ and $-u \in K\,]$ implies $u = 0$.

Note that each nonempty closed cone can be viewed as a direct product of a pointed cone and a linear subspace:

Fact 9. Given a nonempty closed cone K the set $L = K \cap (-K)$ is a linear subspace, $K \cap L^{\perp}$ is a pointed cone and $K = (K \cap L^{\perp}) \oplus L$.

Hint: $L^{\perp}$ is a nonempty closed cone and $\mathbb{R}^n = L^{\perp} \oplus L$ (see [4] § 66).

To derive an important equivalent definition of pointed cone Statement 1 needs be strengthened as follows:

Lemma 2. Let K be a nonempty closed cone and $a \in \mathbb{R}^n \setminus K$. Then there exists $q \in K^*$ such that $\langle q, a \rangle < 0$, $\langle q, u \rangle > 0$ whenever $u \in K \setminus (-K)$, $\langle q, v \rangle = 0$ whenever $u \in K \cap (-K)$.

Proof. Denote $L = K \cap (-K)$.

I. $\forall\ w \in K \cap L^{\perp} \setminus \{0\}\ \ \exists\ \ p \in K^*\ \ \langle p, w \rangle > 0$.
Indeed: As $(-w) \notin K$ use Statement 1 to find $p \in K^*$ with $\langle p, -w \rangle < 0$.

II. $\exists\, x \in K^*\ \ \forall\, w \in K \cap L^{\perp} \setminus \{0\}\ \ \langle x, w \rangle > 0$.
Indeed: Put $L_0 = L^{\perp}$, $U_0 = K \cap L_0$. In case $U_0 = \{0\}$ put $x = 0$. In the opposite case start the following procedure (for $i = 1, 2 \ldots$): supposing $U_{i-1} \neq \{0\}$ choose $w_i \in U_{i-1} \setminus \{0\} \subset K \cap L^{\perp} \setminus \{0\}$, by I find $p_i \in K^*$ with $\langle p_i, w_i \rangle > 0$ and put $L_i = \{v \in L_{i-1};\ \langle p_i, v \rangle = 0\}$ $U_i = K \cap L_i$. As L_i is a proper subspace of L_{i-1}, the dimension of L_i strictly decreases with i (see [4] § 8) and the procedure will stop with $\{0\} = U_k \subset L_k$ for some $k \geq 1$. Consider minimal such k and put $x = p_1 + \ldots + p_k$. By Fact 3 K^* is a cone, hence $x \in K^*$. It makes no problem to verify the required property.

III. $\exists\, x \in K^*\ \ \forall\, u \in K \setminus (-K)\ \ \langle x, u \rangle > 0$.
Indeed: Take $x \in K^*$ from II. Having $u \in K \setminus (-K)$ by Fact 9 write $u = w + v$ where $w \in K \cap L^{\perp}$, $v \in L$. As $w \neq 0$ $\langle x, w \rangle > 0$, as $x \in K^*$ $\langle x, v \rangle = 0$.

IV. $\exists\, q \in K^*\ \ \langle q,a\rangle < 0$, $\langle q,u\rangle > 0$ for $u \in K\setminus(-K)$, $\langle q,v\rangle = 0$ for $v \in K\cap(-K)$. Indeed: Use III. to find the corresponding $x \in K^*$ and Statement 1 to find $p \in K^*$ with $\langle p,a\rangle < 0$. As K^* is a cone (Fact 3) $q_\varepsilon = p + \varepsilon\cdot x \in K^*$ for every $\varepsilon > 0$. Hence $\langle q_\varepsilon, v\rangle = 0$ whenever $v \in K\cap(-K)$ and $\langle q_\varepsilon, u\rangle \geq \varepsilon\langle x,u\rangle > 0$ whenever $u \in K\setminus(-K)$. As $\lim_{\varepsilon\to 0}\langle q_\varepsilon, a\rangle = \langle p,a\rangle < 0$ there exists $\varepsilon > 0$ with $\langle q_\varepsilon, a\rangle < 0$. □

Consequence 2. Having a pointed cone $K \subset \mathbb{R}^n$ for every $a \in \mathbb{R}^n\setminus K$ there exists $q \in K^*$ such that
a) $\langle q,a\rangle < 0$,
b) $\langle q,u\rangle > 0$ whenever $u \in K\setminus\{0\}$.

Proposition 2. Having a nonempty closed cone K the following three conditions are equivalent:
(i) K is pointed
(ii) $\exists\, q \in K^*\ \ \forall\, u \in K\setminus\{0\}\ \ \langle q,u\rangle > 0$
(iii) $\forall\, u \in K\setminus\{0\}\ \ \exists\, p \in K^*\ \ \langle p,u\rangle > 0$.

P r o o f. (i) $\Longrightarrow$ (ii) follows from Consequence 2 (the cone $K = \mathbb{R}^n$ is not pointed for $n \geq 1$), (ii) $\Longrightarrow$ (iii) is trivial, for (iii) $\Longrightarrow$ (i) consider $u \in K\cap(-K)$, supposing $u \neq 0$ find the corresponding $p \in K^*$. But $-u \in K$ implies $\langle p,-u\rangle \geq 0$ and it contradicts $\langle p,u\rangle > 0$. □

6. REGULAR CONES

In this section certain class of closed cones involving pointed cones is introduced. It will be shown later (Proposition 6, § 9) to correspond uniquely to (linear) quasiorderings of integer vectors. Firstly, several technicalities concerning topological properties of dual cones, extreme points and density of $\mathbb{Q}^n$ in linear subspaces are gathered. Then regular cones are defined as cones having $\mathbb{Q}^n$ dense in its boundary subspace. Two equivalent characterization are shown (Proposition 3), the first one says that regular cones are cones having $\mathbb{Q}^n$ dense in their dual cone, the second one characterizes them by certain separation hyperplane theorem. An example of a nonregular cone concludes the section.

Lemma 3. Let K be a nonempty closed cone, denote $L = K\cap(-K)$. Given $q \in K^*$ satisfying $[\langle q,u\rangle > 0$ for $u \in K\setminus(-K)]$ and $a \in \mathbb{R}^n\setminus K$ with $\langle q,a\rangle < 0$, there exists $\varepsilon > 0$ such that $\quad \forall\, p \in L^\perp$ with $\|p-q\| < \varepsilon$ it holds $[\,p \in K^*$ and $\langle p,a\rangle < 0\,]$.

P r o o f. From Fact 9 easily derive $K\setminus(-K) = K\cap L^\perp\setminus\{0\}\oplus L$. Put $S = \{y \in L^\perp;\|y\| = 1\}$. Evidently $\forall\, y \in K\cap L^\perp\setminus\{0\}\ \ \exists\,\alpha > 0\ \ w \in K\cap S\ \ y = \alpha\cdot w$. Similarly, using $\mathbb{R}^n = L^\perp\oplus L$ find $\beta > 0$, $\ s \in S$, $\ v \in L$ with $a = \beta\cdot s + v$. As $K\cap S$ is compact $\gamma = \min\{\langle q,u\rangle;\ \ u \in K\cap S\} > 0$. Put $\varepsilon = \min\{\gamma, |\langle q,s\rangle|\} > 0$. Thus, supposing $p \in L^\perp$ $\|p-q\| < \varepsilon$ it holds $|\langle p,s\rangle - \langle q,s\rangle| < \varepsilon \leq |\langle q,s\rangle|$, i.e. $\langle p,s\rangle < 0$ and hence $\langle p,a\rangle < 0$. Analogously $\langle p,y\rangle > 0$ for all $y \in K\cap L^\perp\setminus\{0\}$ and hence $\langle p,u\rangle > 0$ for all $u \in K\setminus(-K)$. □

Consequence 3. Let K is a nonempty closed cone and $L = K \cap (-K)$.
Then $K^* \subset L^\perp$ and K^* has nonempty interior in $L^\perp$.

Proof. $K^* \subset L^\perp$ is evident. Supposing $K \setminus (-K) \neq \emptyset$ (otherwise $K^* = L^\perp$) take $y \in K \setminus (-K)$, put $a = -y$, by Lemma 2 find $q \in K^*$ with $[\langle q, u\rangle > 0$ for $u \in K \setminus (-K)]$ and with $\langle q, a\rangle < 0$ and apply Lemma 3. □

Definition 5. (extreme point)
A set $C \subset \mathbb{R}^n$ is *convex* iff $[\ \forall\, x, y \in C\ \ \forall\, \alpha \in \langle 0, 1\rangle \quad \alpha \cdot x + (1-\alpha) \cdot y \in C\ ]$.
Given a convex set $C \subset \mathbb{R}^n$ say that $e \in C$ is an *extreme point* of C iff
$\forall\, x, y \in C\ \ [\exists\, \alpha \in (0,1)\ \ e = \alpha \cdot x + (1-\alpha) \cdot y]$ implies $x = y$
(i. e. e is an inner point of none segment in C or equivalently $C \setminus \{e\}$ is convex).
The set of extreme points of C will be denoted by $ex(C)$.

Statement 2. Let $C \subset \mathbb{R}^n$ be a nonempty compact (i. e. closed and bounded) convex set. Then $ex(C) \neq \emptyset$ and C is the convex hull of $ex(C)$, i. e.
$C = \{v \in \mathbb{R}^n;\ v = \sum\limits_{u \in K} \alpha_u \cdot u \text{ where } \emptyset \neq K \subset ex(C) \text{ is finite } \alpha_u \in \mathbb{R}^+\ \sum\limits_{u \in K} \alpha_u = 1\}$.

Comment: This result is well-known as the Minkowski theorem or finite-dimensional version of the Krein-Milman theorem. The reader can it find almost everywhere: in [1] as Theorem 2.13, in [2] as Theorem 5.10, in [7] us Theorem 18.5 or in [6] Theorem 15.1.

Statement 3. Given an $m \times n$ matrix $A = (a_{ij})$ and m-dimensional column vector $b = (b_i)$ denote $P = \{y \in \mathbb{R}^n;\ \sum\limits_{j=1}^{n} a_{ij} \cdot y_j \leq b_i \text{ for all } i = 1, \ldots, m\}$ (polyhedron given by A and b, clearly it is a closed convex set). Let $x \in P$. Then x is an extreme point of P iff there exists $I \subset \{1, \ldots, m\}$ $card\, I = n$ such that the "excised" $n \times n$ matrix $A_I = (a_{ij})_{i \in I}^{j=1,\ldots,n}$ is nonsingular and x is the (unique) solution of the corresponding linear equation system $A_I x = b_I$, i. e. $\forall\, i \in I\ \ \sum\limits_{j=1}^{n} a_{ij} \cdot x_j = b_i$.
Especially: Supposing that all elements of A and b are rational numbers every extreme point of P belongs to $\mathbb{Q}^n$.

Comment: This characterization of extreme points (vertices) of a polyhedron is basics of the familiar linear-programming method for finding all vertices of a polyhedron. We can mention two textbooks where this can be found: in [3] Theorem 18.1 in combination with problem 18.3, in [1] § 4 of the first chapter especially Theorem 2.18. To make sure of the second part of the statement realize that the inverse of a matrix composed of rational numbers is also composed of them. You can either consider the matrix over the field of rational numbers or apply the well-known direct formula for inverse using determinants.

Lemma 4. Let L be a subspace of $\mathbb{R}^n$. Then $\mathbb{Q}^n$ is dense in L (i. e. $\overline{\mathbb{Q}^n \cap L} = L$) iff $\mathbb{Q}^n$ is dense in $L^\perp$ (i. e. $\overline{\mathbb{Q}^n \cap L^\perp} = L^\perp$).

Proof. I. $\mathbb{Q}^n$ is dense in $L \Longrightarrow L$ has a basis made up from elements of $\mathbb{Q}^n$.
Indeed: Take an orthonormal basis $w_1, \ldots, w_k$ of L (for details [4] § 65); choose $\varepsilon > 0$ such that every $k \times k$ matrix $B = (b_{ij})$ is nonsingular whenever

$(\sum_{ij}(b_{ij}-\delta_{ij})^2)^{\frac{1}{2}} < \varepsilon$ (δ_{ij} means Kronecker's delta; to find it realize that considering the matrix norm $\|C\| = (\sum_{ij} c_{ij}^2)^{\frac{1}{2}}$ the determinant is a continous matrix function and use the corresponding nonsingularity characterization – [4] § 53). Find $v_i \in \mathbb{Q}^n \cap L$ with $\|v_i - w_i\| < \varepsilon k^{-\frac{1}{2}}$ and express each $v_i = \sum_{j=1}^{k} a_{ij} \cdot w_j$. By orthonormality of $\{w_j\}$ get

$$\|v_i - w_i\|^2 = \|\sum_{j=1}^{k}(a_{ij} - \delta_{ij}) \cdot w_j\|^2 = \sum_{j=1}^{k}(a_{ij} - \delta_{ij})^2 \quad \text{for} \quad i = 1, \dots, n$$

and hence derive that $A = (a_{ij})$ is nonsingular and $v_1, \dots, v_k$ form a basis.

II. $\mathbb{Q}^n$ is dense in $L^\perp \Longrightarrow \mathbb{Q}^n$ is dense in L.

Indeed: According to I choose a basis $p^1, \dots, p^k \in \mathbb{Q}^n$ of $L^\perp$. It makes no problem to see $L = \{v \in \mathbb{R}^n; \forall\, i = 1, \dots, k \;\langle p^i, v\rangle = 0\}$. Having $w \in L$ and $\varepsilon > 0$ find for each $j = 1, \dots, n$ numbers $a_j, b_j \in \mathbb{Q}$ such that $a_j \leq w_j \leq b_j$ and $b_j - a_j < \varepsilon n^{-\frac{1}{2}}$. Consider the polyhedron $P = \{v \in \mathbb{R}^n; \forall \; j = 1, \dots, n \;\; v_j \leq b_j, \;\; -v_j \leq -a_j$ and $\forall\, i = 1, \dots, k \;\langle p^i, v\rangle \leq 0 \;\langle -p^i, v\rangle \leq 0\}$. As P is bounded and nonempty ($w \in P$) by Statement 2 $ex(P) \neq \emptyset$. By Statement 3 $ex(P) \subset \mathbb{Q}^n$. Thus, take some $u \in ex(P)$. Clearly $u \in L$ and $\|u - w\| < \varepsilon$.

III. $\mathbb{Q}^n$ is dense in $L \Longrightarrow \mathbb{Q}^n$ is dense in $L^\perp$.

Indeed: As $L = L^{\perp\perp}$ ([4] § 62) $\mathbb{Q}^n$ is dense in $(L^\perp)^\perp$ and use II. □

Now, the main definition of this section follows.

Definition 6. (regular cone)
A closed cone $K \subset \mathbb{R}^n$ is called *regular* iff $\mathbb{Q}^n$ is dense in $K \cap (-K)$, i. e. $\overline{\mathbb{Q}^n \cap K \cap (-K)} = K \cap (-K)$.

Evidently, it holds:

Fact 10. Every pointed cone is regular, empty cone is regular.

Fact 11. Having $P \subset \mathbb{R}^n$ such that
a) $u, v \in P \Longrightarrow u + v \in P$ b) $u \in P \;\; \alpha \in \mathbb{Q}^+ \;\; \Longrightarrow \alpha \cdot u \in P$
it holds $\overline{con}(P) = \overline{P}$.

Proposition 3. Let K be a closed cone. Then the following three conditions are equivalent:
(i) K is regular
(ii) $\forall\, a \in \mathbb{R}^n \setminus K \;\; \exists\, p \in \mathbb{Q}^n \cap K^*$ with $\langle p, a\rangle < 0$
(iii) $\mathbb{Q}^n$ is dense in K^* (i. e. $\overline{\mathbb{Q}^n \cap K^*} = K^*$).

Proof. (i) $\Longrightarrow$ (ii)
By Lemma 2 find $q \in K^*$ with $\langle q, a\rangle < 0$ and $[\langle q, u\rangle > 0$ for $u \in K \setminus (-K)]$. Use Lemma 3 to find the corresponding $\varepsilon > 0$ and by (i) and Lemma 4 find $p \in \mathbb{Q}^n \cap L^\perp$ with $\|p - q\| < \varepsilon$.

(ii) $\Longrightarrow$ (iii)
The condition (ii) says $(\mathbb{Q}^n \cap K^*)^* \subset K$. By Fact 4 $K^* \subset (\mathbb{Q}^n \cap K^*)^{**}$ i.e. by Fact 6 and Fact 11 $K^* \subset \overline{con}(\mathbb{Q}^n \cap K^*) = \overline{\mathbb{Q}^n \cap K^*}$, i.e. (iii) was shown.
(iii) $\Longrightarrow$ (i)
By Consequence 3 consider $\emptyset \neq V \subset K^*$ open set in $L^\perp$. By (iii) choose $v \in \mathbb{Q}^n \cap V$ and find $\varepsilon > 0$ such that $T = \{w \in L^\perp;\ \|v - w\| < \varepsilon\} \subset V$. Thus, supposing $u \in L^\perp$ with $\|u\| < \varepsilon$ we have $v - u \in T$ and by (iii) find $w_k \in \mathbb{Q}^n \cap K^*$ with $w_k \to v - u$, i.e. $v - w_k \in \mathbb{Q}^n \cap L^\perp$ converges to u. Any $u \in L^\perp$ can be multiplied by some $\alpha > 0$ to achieve $\|\alpha \cdot u\| < \varepsilon$. Together, $\mathbb{Q}^n$ is dense in $L^\perp$ and by Lemma 4 get (i) . □

Thus, the separation hyperplane theorem for pointed cones can be strengthened as follows.

Consequence 4. Having a pointed cone $K \subset \mathbb{R}^n$, for every $a \in \mathbb{R}^n \setminus K$ there exists $r \in \mathbb{Z}^n$ such that $[\ \forall\, u \in K\ \ \langle r, u\rangle \geq 0\ ]$ and $\langle r, a\rangle < 0$.

Proof. Use Fact 10 and Proposition 3, take $p \in \mathbb{Q}^n \cap K^*$ from (ii) and consider $r = k \cdot p$ where $k \in \mathbb{N}$ ensures $r \in \mathbb{Z}^n$. □

Nevertheless, to illustrate the previous result an example of a nonregular closed cone K such that $\mathbb{Q}^n$ is dense in K is given.

Example. Consider $n = 3$ and put $K = \{(x_1, x_2, x_3);\ \ x_1 \leq \pi x_2\}$, where π is an irrational number. Evidently K is a nonempty closed cone and $\mathbb{Q}^n$ is dense in K but $K \cap (-K) = \{(x_1, x_2, x_3);\ x_1 = \pi x_2\}$ meets $\mathbb{Q}^n$ in $\{(x_1, x_2, x_3);\ x_1 = x_2 = 0\}$.

7. EXTREME RAYS

The concepts of ray and extreme ray are recalled in this section. It is shown that (only) pointed cones have extreme rays and can be determined by means of the set of extreme rays (Proposition 4).

Definition 7. (ray, extreme ray)
Let $x \in \mathbb{R}^n \setminus \{0\}$. The set $R = \{\alpha \cdot x;\ \ \alpha \geq 0\}$ is called the *ray* generated by x. Note that every ray is generated by each its nonzero element.
Supposing K is a nonempty closed cone we say that a ray $R \subset K$ is an *extreme ray of* K iff

$$\forall\, u, v \in K\ \ \frac{1}{2}(u + v) \in R \Longrightarrow u, v \in R. \tag{3}$$

Fact 12. Having a nonempty closed cone K a ray $R \subset K$ is an extreme ray of K iff

$$\forall\, u, v \in K\ \ \ \forall\, \alpha, \beta > 0\ \ \ \alpha \cdot u + \beta \cdot v \in R\ \ \ \Longrightarrow\ \ \ u, v \in R \tag{4}$$

Hint for necessity: Consider $\tilde{u} = 2\alpha \cdot u$, $\tilde{v} = 2\beta \cdot v$, as $\tilde{u}, \tilde{v} \in K$ apply (3).

To derive the above mentioned result the following lemma is needed.

Lemma 5. Let $K \subset \mathbb{R}^n$ be a pointed cone; suppose that $q \in K^*$ satisfies $[\ \langle q, u\rangle > 0$ whenever $u \in K \setminus \{0\}]$ (see Proposition 2). Put $T = \{y \in K;\ \langle q, y\rangle = 1\}$. Then T is a compact convex set. Moreover, given $e \in T$ the following two conditions are equivalent:
(i) e is an extreme point of T
(ii) e generates an extreme ray of K.

Proof. I. T is compact and convex
Indeed: Denote $S = \{y \in \mathbb{R}^n;\ \|y\| = 1\}$, $S' = \{y \in S;\ \langle q, y\rangle > 0\}$, $Q = \{y \in \mathbb{R}^n;\ \langle q, y\rangle = 1\}$ and consider the mapping $t : S' \to Q$ defined by $y \mapsto^t \langle q, y\rangle^{-1} \cdot y$. To verify its continuity realize that $y \mapsto \langle q, y\rangle^{-1}$ is a continuous function. Clearly, $T = t(S' \cap K)$. But $S' \cap K = S \cap K$ is compact (a closed subset of the compact set S) and hence T is compact (for details [5], Chap. 5 Thms. 7, 8).
Moreover, $T = Q \cap K$ implies that T is convex.
II. (i) $\Longrightarrow$ (ii)
Indeed: Suppose $R = \{\alpha \cdot e;\ \ \alpha \geq 0\}$ $u, v \in K$ $\frac{1}{2}(u + v) = \alpha \cdot e$ $\alpha \geq 0$. In case $\alpha = 0$ get $u, -u \in K$ and as K is pointed $u = 0$ and hence $u, v \in R$. Similarly in case $\alpha > 0$ and $[u = 0$ or $v = 0]$. Having $u, v \in K \setminus \{0\}$ and $\alpha > 0$ put $\beta = \langle q, u\rangle$, $x = \beta^{-1} \cdot u$, $\gamma = \langle q, v\rangle$, $y = \gamma^{-1} \cdot v$. Clearly $x, y \in T$ $e = (\frac{1}{2}\alpha^{-1}\beta) \cdot x + (\frac{1}{2}\alpha^{-1}\gamma) \cdot v$ and $1 = \langle q, e\rangle = \frac{1}{2}\alpha^{-1}\beta\langle q, x\rangle + \frac{1}{2}\alpha^{-1}\beta\langle q, y\rangle = \frac{1}{2}\alpha^{-1}\beta + \frac{1}{2}\alpha^{-1}\gamma$. By (i) $x = y = e$, i.e. $u, v \in R$.
III. (ii) $\Longrightarrow$ (i)
Indeed: Suppose $e = \gamma \cdot x + (1-\gamma) \cdot y$ $x, y \in T$ $\gamma \in (0, 1)$. As $x, y \in K$ using Fact 12 and (ii) get $x, y \in \{\alpha \cdot e, \alpha \geq 0\}$. Using $x, y, e \in T$ derive $x = y = e$. □

Proposition 4. Let $\{0\} \neq K$ is a nonempty closed cone. Then K is pointed iff K has extreme rays. Moreover, supposing that $\{0\} \neq K$ is a nonempty pointed closed cone and $L \subset K$ is (any) set generating all its extreme rays it holds $K = con(L)$.

Proof. I. K is not pointed $\Longrightarrow$ K has no extreme rays.
Indeed: Take $u \in K \cap (-K) \setminus \{0\}$, consider a ray R generated by $x \neq 0$ and write $x = \frac{1}{2}(x - u) + \frac{1}{2}(x + u)$. Supposing that R is extreme get $x + u = \beta \cdot x$ for $\beta \geq 0$. But $\beta = 1$ implies $u = 0$ and $\beta \neq 1$ gives $x \in Lin(\{u\})$ i.e. R is generated by u or $(-u)$. But $u = \frac{1}{2}(3u) + \frac{1}{2}(-u)$ implies that R is not extreme.
II. $\{0\} \neq K$ is pointed, $L \subset K$ generates extreme rays $\Longrightarrow L \neq \emptyset$ and $K = con(L)$.
Indeed: Apply Lemma 5. $K \neq \{0\}$ implies $T \neq \emptyset$ and by Statement 2 get $ex(T) \neq \emptyset$. Clearly $\forall\, e \in ex(T)$ $\exists\, u_e \in L$ $\beta_e > 0$ $e = \beta_e \cdot u_e$ by our assumption about L hence $L \neq \emptyset$. By Statement 2 $\forall\, y \in T$ $\exists\, M \subset ex(T)$ finite $y = \sum_{e \in M} \alpha_e \cdot e$ where $\alpha_e \geq 0$ $\sum_{e \in M} \alpha_e = 1$ i.e. $T \subset con(L)$. Hence $K \setminus \{0\} \subset con(L)$ and finally $K = con(L)$. □

We conclude this section by an easy lemma which can be useful when searching extreme rays.

Lemma 6. Suppose that $z : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one-to-one linear mapping. A set $K \subset \mathbb{R}^n$ is a pointed cone iff $z(K)$ is a pointed cone. Having a fixed pointed cone

$K \subset \mathbb{R}^n$ a set R is an extreme ray of K iff $z(R)$ is an extreme ray of $z(K)$.

Proof. As z is continous and $z(K) \cap z(-K) = z(K \cap (-K))$ the first part is easy. Evidently R is the ray generated by an element $x \in \mathbb{R}^n \setminus \{0\}$ iff $z(R)$ is the ray generated by $z(x)$. Finally, it makes no problem to 'shift' the validity of (3) from R to $z(R)$. □

8. PYRAMIDS

Special types of cones, namely pyramids and rational pyramids are studied in this section. Firstly, both concepts are introduced and pointed pyramids are characterized as pointed cones with finite number of extreme rays (Consequence 5). Then pyramids are equivalently characterized as dual cones of finite sets, resp. rational pyramids as dual cones of finite sets of rational vectors (Proposition 5). Hence easily follows that a closed cone is a (rational) pyramid iff its dual cone is a (rational) pyramid and every rational pyramid is a regular cone (Consequence 6, 7). Afterwards, a special separation hyperplane theorem for pointed rational pyramids enabling us to distinguish extreme rays is derived (Consequence 8). The section is concluded by the concept of exhaustive set which is used to characterize sets whose dual cones are pointed rational pyramids (Lemma 8).

Definition 8. (pyramid, rational pyramid)
A set $K \subset \mathbb{R}^n$ is called a *pyramid* iff $K = con(L)$ where $L \subset \mathbb{R}^n$ is finite. If there exists $L \subset \mathbb{Q}^n$ finite such that $K = con(L)$, then K is called a *rational pyramid.*

Note that owing to:

Fact 13. $\forall\, q \in \mathbb{Q}^n \quad \exists\, 0 \neq \beta \in \mathbb{Q}^+ \;\; \exists\, z \in \mathbb{Z}^n \quad q = \beta \cdot z$

It is easy to see:

Fact 14. $K \subset \mathbb{R}^n$ is a rational pyramid iff $K = con(E)$ for finite $E \subset \mathbb{Z}^n$.

The further fact follows from Proposition 1:

Fact 15. Every pyramid is a closed cone.

Proposition 4 implies an easy criterion to recognize whether a pointed cone is a pyramid:

Consequence 5. Let K be a pointed cone. Then
a) K is a pyramid iff K has finitely many extreme rays,
b) K is a rational pyramid iff K has finitely many extreme rays and all of them are generated by elements of $\mathbb{Q}^n$.

Proof. Suppose $K \neq \{0\}$ (otherwise trivial). The sufficiency follows from Proposition 4. For the necessity suppose $K = con(L)$ where L is finite. To show that every extreme ray R (generated by $x \neq 0$) has nonempty intersection with $L \setminus \{0\}$ write $x = (\sum_{u \in L \setminus \{v\}} \alpha_u \cdot u) + \alpha_v \cdot v$ where $\alpha_u \geq 0$, $v \in L \setminus \{0\}$, $\alpha_v > 0$ and by Fact 12 get $v \in R$. As $L \setminus \{0\}$ has finitely many nonempty subsets and the mapping $R \rightarrow R \cap L \setminus \{0\}$ is one-to-one, K has finitely many extreme rays. □

The aim of this section is to prove an important equivalent definition of pyramid for the case that cones are given as dual cones. To show it the following fact will be used.

Fact 16. Supposing K is a pyramid (resp. a rational pyramid) it holds $K^* = L^*$ where $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{Q}^n$) is finite.

Hint: In case $K = \overline{con}(L)$ use Fact 8 to get $K^* = \overline{con}(L)^* = L^*$.

Proposition 5. Suppose $K \subset \mathbb{R}^n$. Then
a) K is a nonempty pyramid iff $K = L^*$ where $L \subset \mathbb{R}^n$ is finite,
b) K is a nonempty rational pyramid iff $K = L^*$ where $L \subset \mathbb{Q}^n$ is finite.

Proof. I. $K = L^*$ where $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{Q}^n$) is finite $\Longrightarrow$ K is a pyramid (resp. rational one).
Indeed: Put $Q = \{x \in \mathbb{R}^n;\ \forall j = 1, \ldots, n \quad -1 \leq x_j \leq 1\}$, $P = K \cap Q$. P can be written as $\{v \in \mathbb{R}^n;\ \forall \ \ r \in L\ \langle -r, v \rangle \leq 0 \quad \forall j = 1, \ldots, n\ \ v_j \leq 1 \quad -v_j \leq 1\}$, i. e. P is a nonempty bounded polyhedron. By Statement 3 the set of extreme points $ex(P)$ is finite (resp. $ex(P) \subset \mathbb{Q}^n$ is finite). Put $T = con(ex(P))$. Evidently $T \subset K$. Conversely, having $u \in K$ find $\alpha > 0$ and $v \in P$ with $u = \alpha \cdot v$. By Statement 2 $v = \sum_{w \in ex(P)} \beta_w \cdot w$ where $\beta_w \geq 0$ $\sum_{w \in ex(P)} \beta_w = 1$ and hence $u \in T$. Thus, $K = T$ i. e. K is a pyramid (resp. rational one).

II. $K \neq \emptyset$ is a pyramid (resp. rational one) $\Longrightarrow$ $K = L^*$ where $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{Q}^n$) is finite.
Indeed: K is a pyramid (resp. rational one) implies by Fact 16 that $K^* = M^*$ where $M \subset \mathbb{R}^n$ (resp. $M \subset \mathbb{Q}^n$) is finite. Using part I get that K^* is a pyramid (resp. rational one). Use Fact 16 once more for K^* to derive $K^{**} = L^*$ where $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{Q}^n$) is finite. But Fact 15 and Consequence 1 imply $K = K^{**}$. □

Note that the result saying that every pyramid is a dual cone of a finite set proved in Proposition 5a is very old (it is an easy consequence of the main theorem from [9]).

Consequence 6. Suppose that $K \subset \mathbb{R}^n$ is a closed cone. Then it holds:
a) K is a pyramid iff K^* is a pyramid,
b) K is a rational pyramid iff K^* is a rational pyramid.

Proof. The necessity follows from Fact 16 and Proposition 5. In case $K \neq \emptyset$ the necessity also yields the sufficiency by means of Consequence 1 ($K = K^*$). □

Consequence 7. Every rational pyramid is a regular cone.

Proof. Consider a rational pyramid K, by Consequence 6 $K^* = con(L)$ where $L \subset \mathbb{Q}^n$ is finite. Put $V = \{ \sum_{t \in L} \beta_t \cdot t \ ; \ \beta_t \in \mathbb{Q}^+ \}$, evidently $\overline{V} = K^*$ i. e. $\mathbb{Q}^n$ is dense in K^* and by Proposition 3 K is regular. □

Lemma 7. Let $K \subset \mathbb{R}^n$ be a pointed pyramid and R its extreme ray. Then there exists $t \in \mathbb{R}^n$ such that $[\langle t, x\rangle = 0 \quad \text{for every } x \in R]$ and $[\langle t, s\rangle > 0 \quad \text{for all } s$ generating the other extreme rays $]$.

Proof. By Proposition 2 find $q \in \mathbb{R}^n$ with $[\langle q, u\rangle > 0$ for $u \in K \setminus \{0\}\,]$ and put $T = \{y \in K; \langle q, y\rangle = 1\}$. Take the uniquely determined $r \in R \cap T$ (see Lemma 5) and put $\tilde{K} = con(E \setminus \{r\})$ where E denotes the set of extreme points of T. In case $E \setminus \{r\} \neq \emptyset$ by Statement 1 find $p \in \tilde{K}^*$ with $\langle p, r\rangle < 0$. It makes no problem to see that $t = p - \langle p, r\rangle \cdot q$ satisfies both $\langle t, r\rangle = 0$ and $\langle t, e\rangle \geq -\langle p, r\rangle > 0$ for all $e \in E \setminus \{r\}$. The rest follows from Lemma 5. □

Consequence 8. Let $K \subset \mathbb{R}^n$ be a pointed rational pyramid and R its extreme ray. Then there exists $q \in \mathbb{Q}^n$ such that $[\langle q, r\rangle = 0 \quad \text{for all } r \in R]$ and $[\langle q, x\rangle > 0$ for all x generating the other rays of $K]$.

Proof. Let r_0 generates R, by Consequence 5b $\mathbb{Q}^n$ is dense in $L = \{\alpha \cdot r_0 \ ; \ \alpha \in \mathbb{R}\}$ and by Lemma 4 $\mathbb{Q}^n$ is dense in $L^\perp$. By Lemma 7 there exists $t \in L^\perp$ such that $\langle t, s\rangle > 0$ for all $s \in S$ where S denotes the set of points s generating the remaining extreme rays and specified by the requirement $\|s\| = 1$. Find $q \in \mathbb{Q}^n \cap L^\perp$ with $\|q - t\| < \min\{\langle s, t\rangle \ ; \ s \in S\}$ (S is finite!) and the inequality $|\langle s, t\rangle - \langle s, q\rangle| \leq \|s\| \cdot \|q - t\| \leq \langle s, t\rangle$ implies $\langle s, q\rangle > 0$ for each $s \in S$. Whenever x generates another ray of K write $x = \sum_{s \in S} \alpha_s \cdot s + \alpha_r \cdot r$ where $\alpha_i \in \mathbb{R}^+$, necessarily $\alpha_s > 0$ for some $s \in S$ and hence $\langle q, x\rangle \geq \alpha_s \cdot \langle q, s\rangle > 0$. □

Note that the preceding result does not hold for general pointed cones:

Example. Consider $n = 3$ and put $K = con(L)$ where
$L = \{ (x_1, x_2, x_3) \ ; \ {x_1}^2 + {x_2}^2 \leq 1 \quad x_3 = 1 \} \cup \{ (-1, -1, 1), (-1, 1, 1) \}$
(the base of this cone is a circle with an attached oblong). Consider the ray generated by $r = (0, 1, 1)$. The only $q \in K^*$ with $\langle q, r\rangle = 0$ satisfies $\langle q, x\rangle = 0$ for $x = (-1, 1, 1)$ generating another extreme ray of K.

Definition 9. (exhaustive set)
A set $E \subset \mathbb{R}^n$ is called *exhaustive* iff $E^* \cap (-E^*) \cap \mathbb{Z}^n = \{0\}$, i. e.
$\forall z \in \mathbb{Z}^n \quad [\, \forall e \in E \ \ \langle z, e\rangle = 0 \,] \quad \Longrightarrow \quad z = 0.$

Lemma 8. a) Whenever $E \subset \mathbb{Z}^n$ is finite exhaustive it holds $E^* \cap (-E^*) = \{0\}$.
b) Supposing that $K \subset \mathbb{R}^n$ is a closed cone the following conditions are equivalent:
(i) $K = con(E)$ where $E \subset \mathbb{Z}^n$ is a finite exhaustive set
(ii) K^* is a pointed rational pyramid.

P r o o f. a) Put $K = con(E)$; it is a rational pyramid. By Fact 8 $K^* = E^*$. Using Consequences 6b and 7 derive that K^* is a regular cone. The exhaustivity assumption says $K^* \cap (-K^*) \cap \mathbb{Z}^n = \{0\}$, hence by Fact 13 $\{0\} = \overline{K^* \cap (-K^*) \cap \mathbb{Q}^n}$ and $\{0\} = \overline{K^* \cap (-K^*) \cap \mathbb{Q}^n} = K^* \cap (-K^*) = E^* \cap (-E^*)$.
b) The implication (i) $\Longrightarrow$ (ii) has been already proved above; supposing (ii) by Consequence 6b K is a rational pyramid, hence by Fact 14 $K = con(E)$ for some finite $E \subset \mathbb{Z}^n$; Fact 8 says $K^* = E^*$ and hence $E^* \cap (-E^*) \cap \mathbb{Z}^n \subset K^* \cap (-K^*) = \{0\}$ gives the exhaustivity. □

9. QUASIORDERINGS OF INTEGER VECTORS

The focus of study of this section are restrictions of closed cones (and hence the corresponding quasiorderings) to $\mathbb{Z}^n$ – the class of integer vectors. Firstly, intersections of $\mathbb{Z}^n$ with (general) closed cones are characterized (Lemma 9). Further result (Proposition 6) identifies them with regular cones and shows that the correspondence is antitonne.

The method of establishing of quasiorderings (see the 9.Introduction) for integer vectors leads to the concept of cover (Definition 10, in fact the set of conical combinations with rational coefficients). The cover is shown to be equal to intersection of $\mathbb{Z}^n$ with conic hull (Lemma 10). On the other hand, the method of inducing of quasiorderings for integer vectors suggests to intersect $\mathbb{Z}^n$ with a dual cone. Proposition 7 says that quasiordering on $\mathbb{Z}^n$ is finitely inducable iff it coincides with the cover of a finite set or iff it is given by a rational pyramid. In the second part of this proposition the existence of the least set of normalized integer vectors inducing such quasiorderings is proved in certain special case, namely that the quasiordering is established by an exhaustive set. Moreover, elements of this least inducing set can be distinguished by integer vectors which are positive with respect to the quasiordering. Certain method to achieve elements of this least inducing set is indicated by Lemma 12.

Finally, Proposition 8 characterizes similarly orderings given by rational pyramids. The existence of the least set of normalized integer vectors establishing such orderings is proved there too.

Lemma 9. Supposing $L \subset \mathbb{Z}^n$ the following two conditions are equivalent:
(i) $L = K \cap \mathbb{Z}^n$ for some nonempty closed cone K
(ii) L satisfies the following three conditions:

$$u, v \in L \implies u + v \in L \qquad (\beta.1)$$
$$u_k \in L \;\; u \in \mathbb{Z}^n \;\; \beta_k, \beta \in \mathbb{N} \;\; \beta_k^{-1} \cdot u_k \to \beta^{-1} \cdot u \implies u \in L \qquad (\beta.2)$$
$$0 \in L \qquad (\beta.3)$$

P r o o f. The implication (i) $\Longrightarrow$ (ii) is easy to see. To show (ii) $\Longrightarrow$ (i) put $P = \{\alpha \cdot v;\; v \in L \;\; \alpha \in \mathbb{Q}^+\}$. Using ($\beta$.1) and Fact 11 get $\overline{P} = \overline{con}(P)$. It remains to see $L = \overline{P} \cap \mathbb{Z}^n$. Clearly, $L \subset \overline{P} \cap \mathbb{Z}^n$. Conversely, take $u \in \overline{P} \cap \mathbb{Z}^n$ and consider $u^k \in L$, $\alpha_k \in \mathbb{Q}^+$ with $\alpha_k \cdot u^k \to u$. In case $u = 0$ use (β.3). Supposing $u \neq 0$ find a nonzero component $u_j \neq 0$ of u $(j \in \{1, \ldots, n\})$. For large indices k the numbers

$(u^k)_j$ have the same sign as u_j. In case $u_j > 0$ put $\beta_k = (u^k)_j$ $\beta = u_j$ (otherwise $\beta_k = -(u_k)_j$, $\beta = -u_j$) and as $\alpha_k\beta_k \to \beta > 0$ use $(\beta.2)$ to derive $u \in L$. □

Intersections of $\mathbb{Z}^n$ with closed cones can be understood as quasiorderings on $\mathbb{Z}^n$. The following result identifies them with regular cones. Note that by Consequence 1 K is a nonempty closed cone iff it has the form B^* for some $B \subset \mathbb{R}^n$.

Proposition 6. a) Whenever $L = \mathbb{Z}^n \cap B^*$ where $B \subset \mathbb{R}^n$ then $K = (\mathbb{Z}^n \cap B^*)^*$ is a regular cone satisfying $L = \mathbb{Z}^n \cap K^*$.

b) Whenever K_1, K_2 are regular cones, then $\mathbb{Z}^n \cap K_1^* \subset \mathbb{Z}^n \cap K_2^*$ is equivalent with $K_2 \subset K_1$. Especially, the regular cone mentioned in a) is uniquely determined.

Proof. a) Clearly by Fact 5 $\mathbb{Z}^n \cap B^* \subset (\mathbb{Z}^n \cap B^*)^{**} = K^*$. Conversely, by Fact 4 and Fact 7 $\mathbb{Z}^n \cap K^* \subset K^* = (\mathbb{Z}^n \cap B^*)^{**} \subset B^{***} = B^*$. Thus $\mathbb{Z}^n \cap B^* = \mathbb{Z}^n \cap K^*$. Hence, by Fact 4, Fact 6 and Fact 11
$K^* = (\mathbb{Z}^n \cap B^*)^{**} \subset (\mathbb{Q}^n \cap K^*)^{**} = \overline{con}(\mathbb{Q}^n \cap K^*) = \overline{\mathbb{Q}^n \cap K^*}$, i.e. K is a regular cone by Fact 3 and Proposition 3(iii).

b) By Fact 4 $K_2 \subset K_1$ implies $\mathbb{Z}^n \cap K_1^* \subset \mathbb{Z}^n \cap K_2^*$. Conversely, supposing $\mathbb{Z}^n \cap K_1^* \subset \mathbb{Z}^n \cap K_2^*$ consider $x \in K_2 \setminus K_1$. By Proposition 3(ii) find $p \in \mathbb{Q}^n \cap K_1^*$ with $\langle p, x\rangle < 0$. Using Fact 13 find $z \in \mathbb{Z}^n \cap K_1^*$ with $\langle z, x\rangle < 0$ and this contradicts the assumption. □

To characterize rational pyramids in $\mathbb{Z}^n$ the following concept of cover will be used.

Definition 10. (cover)
Let $L \subset \mathbb{Z}^n$. Introduce its *cover* denoted by $cov(L)$ as follows:
$cov(L) = \{u \in \mathbb{Z}^n;\ u = \sum_{v \in K} \beta_v \cdot v$ where $\emptyset \neq K \subset L$ is finite $\beta_v \in \mathbb{Q}^+\}$.

Fact 17. Having $L \subset \mathbb{Z}^n$ it holds
$cov(L) = \{u \in \mathbb{Z}^n;\ k \cdot u = \sum_{v \in K} \lambda_v \cdot v$ where $\emptyset \neq K \subset L$ is finite $k \in \mathbb{N}$ $\lambda_v \in \mathbb{Z}^+\}$.

Lemma 10. Suppose that $L \subset \mathbb{Z}^n$.
a) Then $cov(L) = con(L) \cap \mathbb{Z}^n$.
b) If moreover $\emptyset \neq L$ is finite, then $cov(L) = \mathbb{Z}^n \cap L^{**}$.

Proof. Clearly $cov(L) \subset con(L) \cap \mathbb{Z}^n$. Conversely, having $u \in con(L) \cap \mathbb{Z}^n$ write $u = \sum_{i=1}^{k} \beta_i \cdot v^i$ where $v^i \in L$ $\beta_i \in \mathbb{R}^+$ $i = 1, \ldots, k$. Fix the vector $(\beta_1, \ldots, \beta_k)$ and find $(\gamma_1, \ldots, \gamma_k)$, $(\delta_1, \ldots, \delta_k) \in \mathbb{Q}^k$ such that $0 \leq \gamma_i \leq \beta_i \leq \delta_i$ for $i = 1, \ldots, k$ and put:

$$P = \{(x_1, \ldots, x_k) \in \mathbb{R}^k;\ \forall\, i = 1, \ldots, k \quad x_i \leq \delta_i \quad -x_i \leq -\gamma_i$$
$$\forall\, j = 1, \ldots, n \quad \sum_{i=1}^{k} x_i (v^i)_j \leq u_j \quad \sum_{i=1}^{k} x_i (-v^i)_j \leq -u_j\}.$$

As $(\beta_1, \ldots, \beta_k) \in P$ it is a nonempty bounded polyhedron. By Statement 2 P has an extreme point $(\alpha_1, \ldots, \alpha_k)$ and by Statement 3 $(\alpha_1, \ldots, \alpha_k) \in \mathbb{Q}^k$. But $(\alpha_1 \ldots, \alpha_k) \in$

P means $u = \sum_{i=1}^{k} \alpha_i \cdot v_i$ i. e. $u \in cov(L)$.
b) By Proposition 1 and Fact 6 $con(L) = \overline{con}(L) = L^{**}$. □

To ensure the uniqueness of the least set of establishing (resp. inducing) integer vectors the following concept is needed.

Definition 11. (normalized integer vector)
Denote by $\mathbb{Z}^n_{norm}$ the class of all vectors $u \in \mathbb{Z}^n$ such that the collection of its components $u_1, ..., u_n$ has no common prime divisor (especially $0 \notin \mathbb{Z}^n_{norm}$).

We mention several facts about $\mathbb{Z}^n_{norm}$:

Fact 18. $\forall\, z \in \mathbb{Z}^n \setminus \{0\} \quad \exists\, m \in \mathbb{N} \quad \overline{z} \in \mathbb{Z}^n_{norm} \quad z = m \cdot \overline{z}$.

Fact 19. $\forall\, z_1, z_2 \in \mathbb{Z}^n_{norm} \quad [\, k \cdot z_1 = l \cdot z_2 \text{ for } k, l \in \mathbb{N}\,] \quad \Longrightarrow \quad z_1 = z_2$.

Fact 20. Every ray contains at most one element of $\mathbb{Z}^n_{norm}$.

Lemma 11. Suppose that $E \subset \mathbb{Z}^n$ is finite and exhaustive with $E^* \neq \{0\}$.
a) The following two conditions are equivalent for $K \subset \mathbb{R}^n$:
(i) $\overline{con}(K) = E^*$
(ii) $cov(E) = \mathbb{Z}^n \cap K^*$.
b) Define A as the set of all elements $a \in \mathbb{Z}^n \cap E^* \setminus \{0\}$ satisfying

$$[\, k \cdot a = z_1 + z_2 \quad k \in \mathbb{N} \quad z_1, z_2 \in \mathbb{Z}^n \cap E^*\,] \quad \Longrightarrow \quad [\, \exists\, l \in \mathbb{Z}^+ \quad z_1 = l \cdot a\,] \tag{5}$$

Then the set A is finite and it is the least subset of $\mathbb{Z}^n_{norm}$ such that $con(A) = E^*$.

Proof. a) I. In case $K = \emptyset$ (i) is untrue by Consequence 1 and (ii) by Fact 8 contradicts the assumption $E^* \neq \{0\}$. Thus suppose $K \neq \emptyset$ in the sequel.
II. (i) $\Longrightarrow$ (ii)
By Fact 8, (i), Fact 6 and Proposition 1 write $K^* = \overline{con}(K)^* = E^{**} = \overline{con}(E) = con(E)$. Then use Lemma 10a.
III. (ii) $\Longrightarrow$ (i)
Certainly $E \subset cov(E) \subset K^*$ implies by Fact 4 $K^{**} \subset E^*$. Thus by Fact 6 the first inclusion $\overline{con}(K) \subset E^*$ is shown. By Lemma 8a $\{0\} \subset K^{**} \cap (-K^{**}) \subset E^* \cap (-E^*) = \{0\}$ derive that K^{**} is pointed and therefore by Fact 10 regular. By Proposition 3(iii) and Fact 7 get $K^* = \overline{\mathbb{Q}^n \cap K^*}$. Nevertheless (ii) says $\mathbb{Z}^n \cap K^* \subset cov(E) \subset \overline{con}(E)$ hence (Fact 13) $\mathbb{Q}^n \cap K^* \subset \overline{con}(E)$ and further $K^* = \overline{\mathbb{Q}^n \cap K^*} \subset \overline{con}(E)$. Hence by Fact 8, Fact 4 and Fact 6 $E^* = \overline{con}(E)^* \subset K^{**} = \overline{con}(K)$.
b) IV. $A \subset \mathbb{Z}^n_{norm}$.
Whenever $a \in A$ then by Fact 18 $a = m \cdot \overline{a}$ for $m \in \mathbb{N}$, $\overline{a} \in \mathbb{Z}^n_{norm}$. Put $k = 1$, $z_1 = \overline{a}$, $z_2 = (m-1) \cdot \overline{a}$ in (5) and derive $\overline{a} = z_1 = l \cdot a = lm \cdot \overline{a}$ for $l \in \mathbb{Z}^+$. Necessarily $l \in \mathbb{N}$ and hence $l = m = 1$ says $a = \overline{a}$.
V. There exists finite $K \subset \mathbb{Z}^n_{norm}$ with $con(K) = E^*$.
By Proposition 5 and Fact 14 $E^* = con(T)$ for finite $T \subset \mathbb{Z}^n$. Using Fact 18 for each $z \in T \setminus \{0\}$ find $\beta_z \in \mathbb{N}$ and $\overline{z} \in \mathbb{Z}^n_{norm}$ with $z = \beta_z \cdot \overline{z}$; then put $K = \{\overline{z};\ z \in T \setminus \{0\}\,\}$.

VI. Whenever $K \subset \mathbb{Z}^n_{norm}$ with $con(K) = E^*$ then $A \subset K$.
By Lemma 10 $\mathbb{Z}^n \cap E^* = \mathbb{Z}^n \cap con(K) = cov(K)$, hence by Fact 17 every $a \in A$ can be decomposed: $k \cdot a = \sum_{v \in K'} \gamma_v \cdot v$ for $\emptyset \neq K' \subset K \quad \gamma_v \in \mathbb{Z}^+ \quad k \in \mathbb{N}$. As $a \neq 0$ there exists $v \in K'$ with $\gamma_v \in \mathbb{N}$; put $z_1 = v \quad z_2 = k \cdot a - v$, evidently $z_i \in \mathbb{Z}^n \cap E^*$ and by (5) $z_1 = l \cdot a$ for some $l \in \mathbb{Z}^+$. Clearly $l \in \mathbb{N} \;\; (0 \notin \mathbb{Z}^n_{norm})$ and hence by Fact 19 and IV $\quad z_1 = a$.

VII. If [$K \subset \mathbb{Z}^n_{norm}$ is finite with $E^* = con(K) \quad z \in K \setminus A$] then $E^* = con(K \setminus \{z\})$.
Clearly $z \in E^* \cap \mathbb{Z}^n \setminus \{0\}$. As $z \notin A$ find $k \in \mathbb{N} \quad z_i \in \mathbb{Z}^n \cap E^*$ such that $[k \cdot z = z_1 + z_2 \;\;\&\;\; \forall\, l \in \mathbb{Z}^+ \; z_1 \neq l \cdot z]$. By Lemma 10 $\;\mathbb{Z}^n \cap E^* = \mathbb{Z}^n \cap con(K) = cov(K)$ and therefore by Fact 17 write $k^i \cdot z_i = \sum_{v \in K} \gamma^i_v \cdot v$ for $k^i \in \mathbb{N},\; \gamma^i_v \in \mathbb{Z}^+ \quad (i = 1, 2)$
Hence easily get $k\,k^1\,k^2 \cdot z = \sum_{v \in K} (k^1\,\gamma^2_v + k^2\,\gamma^1_v) \cdot v$. In case $kk^1k^2 \leq k^1\gamma^2_z + k^2\gamma^1_z$ simply get $\forall\, v \in K \setminus \{z\} \quad \gamma^1_v \cdot v \in E^* \cap (-E^*) \cap \mathbb{Z}^n$ and therefore (the exhaustivity) $\gamma^1_v = 0$. Thus $k^1 \cdot z_1 = \gamma^1_z \cdot z$ and by Fact 19 it contradicts $[\forall\, l \in \mathbb{Z}^n \; z_1 \neq l \cdot z]$. Therefore $kk^1k^2 > k^1\gamma^2_z + k^2\gamma^1_z$ and $z \in con(K \setminus \{z\})$. Hence $K \subset con(K \setminus \{z\})$ says $E^* = con(K) \subset con(K \setminus \{z\}) \subset con(K)$.
VIII. By V find $K \subset \mathbb{Z}^n_{norm}$ finite with $con(K) = E^*$. By VII remove all elements of $K \setminus A$ saving $con(K) = E^*$. Owing to VI exactly A remains. Thus A is finite and satisfies $con(A) = E^*$. The rest follows from VI. □

Proposition 7.
a) Let $L \subset \mathbb{Z}^n$. Then the following three conditions are equivalent :
(i) $L = \mathbb{Z}^n \cap K$ where $\emptyset \neq K \subset \mathbb{R}^n$ is a rational pyramid
(ii) $L = cov(E)$ for finite $\emptyset \neq E \subset \mathbb{Z}^n$
(iii) $L = \mathbb{Z}^n \cap M^*$ for $M \subset \mathbb{Z}^n$ finite.
b) Supposing that $E \subset \mathbb{Z}^n$ is finite and exhaustive there exists the least finite $A \subset \mathbb{Z}^n_{norm}$ such that $cov(E) = \mathbb{Z}^n \cap A^*$. Moreover, it holds

$$\forall\, a \in A \;\; \exists\, u \in \mathbb{Z}^n \;\; \langle a, u\rangle = 0 \;\;\&\;\; [\forall\, s \in A \setminus \{a\} \;\; \langle s, u\rangle > 0] \tag{6}$$

P r o o f. a) (i) $\Longleftrightarrow$ (ii) easily follows from Lemma 10a and Fact 14, (i) $\Longleftrightarrow$ (iii) is an easy consequence of Proposition 5b and Fact 13.

b) The statement is easy in case $E^* = \{0\}$: by Lemma 10b $cov(E) = \mathbb{Z}^n \cap E^{**} = \mathbb{Z}^n$, it suffices to take $A = \emptyset$. Thus, suppose $E^* \neq \{0\}$ and by Lemma 11b take the least set $A \subset \mathbb{Z}^n_{norm}$ with $con(A) = E^*$. By Proposition 1 and Lemma 11a get $cov(E) = \mathbb{Z}^n \cap A^*$. Whenever $M \subset \mathbb{Z}^n_{norm}$ is finite with $cov(E) = \mathbb{Z}^n \cap M^*$ by the same argument get $con(M) = \overline{con}(M) = E^*$ and hence $A \subset M$ by Lemma 11b.
Further, by Proposition 5b E^* is a rational pyramid and by Lemma 8 a pointed cone. Therefore by Consequence 5b every of finite number of its extreme rays is generated by an element of $\mathbb{Q}^n$ and thus by Facts 13, 18, 20 by the unique element of $\mathbb{Z}^n_{norm}$; denote this finite subset of $\mathbb{Z}^n_{norm}$ by B. Proposition 4 says $E^* = con(B)$ and by Lemma 11b $A \subset B$. Having $a \in A \subset B$ apply Consequence 8 to find $q \in \mathbb{Q}^n$ with $\langle q, a\rangle = 0$ and $[\forall\, \overline{a} \in A \setminus \{a\} \;\; \langle q, \overline{a}\rangle > 0]$, then use Fact 13. □

Having a concrete finite exhaustive $E \subset \mathbb{Z}^n$ you can sometimes face the problem to find the least finite $A \subset \mathbb{Z}^n_{norm}$ with $con(E) = \mathbb{Z}^n \cap A^*$ (see Proposition 7b). The characterization of A from Lemma 11b (namely the condition (5)) is too clumsy for this purpose. Below a more convenient equivalent definition is given.

Definition 12. (portrait)
Having finite exhaustive $E \subset \mathbb{Z}^n$ with $E^* \neq \{0\}$ for each $s \in E^*$ introduce its *portrait* E_s *(in* E*)* as follows: $E_s = \{u \in E\,;\, \langle s, u\rangle > 0\}$.

Lemma 12. Suppose that $E \subset \mathbb{Z}^n$ is finite and exhaustive with $E^* \neq \{0\}$. An element $a \in \mathbb{Z}^n_{norm} \cap E^*$ satisfies the condition (5) from Lemma 11b iff its portrait in E is minimal within $\mathbb{Z}^n_{norm} \cap E^*$ i. e. it holds:

$$\forall\, s \in \mathbb{Z}^n_{norm} \cap E^* \quad E_s \subset E_a \quad \Longrightarrow \quad E_s = E_a \tag{7}$$

Proof. I. $\forall\, r, s \in \mathbb{Z}^n \cap E^* \quad E_s \subset E_r \quad \Longleftrightarrow \quad [\exists\, k \in \mathbb{N} \;\; k \cdot r - s \in E^*]$.
The sufficiency is trivial, for necessity find (for each $u \in E$) $k_u \in \mathbb{N}$ with $k_u \cdot \langle r, u\rangle \geq \langle s, u\rangle$ and put $k = \max\{k_u; u \in E\}$.

II. $a \in A \quad \Longrightarrow \quad [\forall\, s \in \mathbb{Z}^n_{norm} \cap E^* \;\; E_s \subset E_a \implies s = a] \quad \Longrightarrow \quad (7)$.
By I find $k \in \mathbb{N}$ with $k \cdot a - s \in E^*$ and put $z_1 = s \;\; z_2 = k \cdot a - s$; as $a \in A$ by (5) $s = z_1 = l \cdot a$ for $l \in \mathbb{Z}^+$, necessarily $l \in \mathbb{N}$ and by Fact 19 $s = a$.

III. $\forall\, s \in \mathbb{Z}^n_{norm} \cap E^* \quad \exists\, a \in A \quad E_a \subset E_s$.
By Lemma 11b $E^* = con(A)$, i. e. $\mathbb{Z}^n \cap E^* = cov(A)$ by Lemma 10a, decompose s as suggested in Fact 17, choose $a \in A$ with nonzero coefficient and apply I.

IV. $a \in \mathbb{Z}^n_{norm} \cap E^*$ satisfies (7) $\quad \Longrightarrow \quad a \in A$.
Using III find $b \in A$ with $E_b \subset E_a$. Owing to (7) $E_b = E_a$ and by II (take b instead of a) get $a = b$, i. e. $a \in A$. □

The corresponding version of Proposition 7 for pointed rational pyramids follows.

Proposition 8. Let $L \subset \mathbb{Z}^n$.
a) Then the following four conditions are equivalent :
- (i) $L = K \cap \mathbb{Z}^n$ where $K \neq \emptyset$ is a pointed rational pyramid
- (ii) $L = cov(A)$ for finite $\emptyset \neq A \subset \mathbb{Z}^n$ such that $[\exists\, q \in \mathbb{R}^n \;\; \langle q, u\rangle > 0$ for $u \in A \setminus \{0\}]$
- (iii) $L \cap (-L) = \{0\}$ and $L = cov(A)$ for finite $\emptyset \neq A \subset \mathbb{Z}^n$
- (iv) $L = \mathbb{Z}^n \cap M^*$ where $M \subset \mathbb{Z}^n$ is finite and exhaustive.

b) Whenever any of preceding conditions is satisfied there exists the least subset $E \subset \mathbb{Z}^n_{norm}$ satisfying $L = cov(E)$ (naturally finite by (iii)).

Proof. a) (i) $\Longrightarrow$ (ii)
By Fact 14 and Fact 1 $K = con(A)$ with finite $\emptyset \neq A \subset \mathbb{Z}^n$. As K is pointed by Proposition 2 there exists $q \in \mathbb{R}^n$ such that $[\langle q, u\rangle > 0$ whenever $u \in K \setminus \{0\}]$. Then use Lemma 10a.
(ii) $\Longrightarrow$ (iii) is evident as $\langle q, u\rangle > 0$ whenever $u \in cov(A) \setminus \{0\}$.
(iii) $\Longrightarrow$ (iv) use Proposition 7a and $M^* \cap (-M^*) \cap \mathbb{Z}^n \subset L \cap (-L)$.
(iv) $\Longrightarrow$ (i) By Lemma 8b M^* is a pointed rational pyramid.

b) By Consequence 5b every of finite number of extreme rays of K intersects $\mathbb{Q}^n$ and therefore by Facts 13, 18 also $\mathbb{Z}^n_{norm}$, by Fact 20 this element is unique. Define E as the set of these elements, by Proposition 4 $K = con(E)$ and using Lemma 10a $L = cov(E)$. Moreover, having $\tilde{E} \subset \mathbb{Z}^n_{norm}$ with $L = cov(\tilde{E})$ consider $x \in E$ and write

$x = \sum_{v \in F} \beta_v \cdot v + \beta_u \cdot u$ where $F \subset \tilde{E}$ is finite, $u \in \tilde{E} \setminus \{0\}$, $\beta_v \in \mathbb{Q}^+$, $0 < \beta_u \in \mathbb{Q}$.

Consider the ray R generated by x, using Fact 12 $u \in R$, by Fact 20 $x = u$, i.e. $x \in \tilde{E}$. Therefore $E \subset \tilde{E}$. □

10. CONCLUSION

Let us give a short summary of the main results of the paper.

Proposition 2 gives an equivalent definition of pointed cones, similarly Proposition 3 gives equivalent definitions of regular cones and Proposition 5 equivalent definitions of pyramids and rational pyramids. Note that every rational pyramid is a regular cone according to Consequence 7.

Regular cones are shown in Proposition 6 to correspond to quasiorderings of integer vectors. Two equivalent definitions of such orderings corresponding to rational pyramids are derived in Proposition 7. Moreover, the existence of a uniquely determined finite set inducing this ordering is proved and "separation property" of this set shown. Especially these results are utilized in [8].

Proposition 8 gives equivalent definitions of orderings of integer vectors corresponding to rational pyramids and shows that they can be established by means of a uniquely determined finite set of normalized integer vectors.

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