M. Studený Derivování měr v metrických prostorech (in Czech, translation: Differentiation of measures in metric spaces), graduate diploma thesis, Faculty of Mathematics and Physics, Charles University, Prague 1981, Czechoslovakia, 115 pages.

ABSTRACT Let M be a separable metric space, $U(x,r)$ denotes the closed ball with center $x \in M$ and radius $r > 0$, and μ, ν are locally finite Borel measures on M. The *differentiation theorems* concern the behaviour of the ratios $\nu(U(x,r))/\mu(U(x,r))$ when r tends to zero, in the sense whether it converges to the Radon-Nikodym derivative $d\nu_{ac}/d\mu\,(x)$ of the absolutely continous part of ν with respect to μ.

The differentiation theorems are classified according to two criteria. The first criterion is the class of measures ν which are considered for a fixed measure μ. One can distinguish the differentiation theorem for all locally finite measures, for measures absolutely continuous with respect to μ (= integral differentiation theorem), for measures singular with respect to μ, and for measures which are restrictions of μ to a certain Borel set $A \subset M$ (= density theorem). The second criterion is the type of convergence. One can consider the convergence μ-almost everywhere or the convergence in measure μ (on sets of finite measure μ). Of course, the convergence μ-almost everywhere implies the covergence in measure μ then. A special 'inequality-type' theorem (implied by the theorem with the convergence in measure μ) is the requirement that the value of $d\nu_{ac}/d\mu$ is μ-almost everywhere closed between the values of $\liminf_{r\to 0} \nu(U(x,r))/\mu(U(x,r))$ and $\limsup_{r\to 0} \nu(U(x,r))/\mu(U(x,r))$. The last type, a 'week inequality-type' theorem is obtained by a further modification, that is, for a fixed $x \in M$ and $r > 0$ one considers the ratios $\nu(U(y,s))/\mu(U(y,s))$ for all balls $U(y,s)$ contained in $U(x,r)$ (not only balls with the center x as in the case of the ordinary inequality-type theorem).

In starting chapters miscellaneous equivalent formulations of differentiation theorems are given and compared. The third chapter shows that the validity of the differentiation theorems is saved when μ is replaced by an equivalent measure $\bar{\mu}$ in sense that $\bar{\mu}$ is absolutely continous with respect to μ and has a continous strictly positive Radon-Nikodym derivative. *Universal differentiation theorem*, that is, the differentiation theorems valid for all locally finite Borel measures μ on M are dealt with in the fourth chapter. Several equivalent formulations of these theorems are derived. For example, the universal weak inequality-type theorem easily follows from the following *comparison principle*: if there exist $r_0 > 0$ such that the inequality $\nu(U(x,r)) \leq \mu(U(x,r))$ holds for all $x \in M$ and $0 < r \leq r_0$, then $\nu(A) \leq \mu(A)$ for every Borel set $A \subset M$.

We say that a non-zero locally finite Borel measure μ on M is *uniform* if $\mu(U(x,r))$ does not depend on x. It is *almost uniform* if there exist $0 < c \leq 1$ and a nondecreasing function $h : (0,\infty) \to (0,\infty)$ with $\lim_{r\to 0} h(r) = 0$ such that $c \cdot h(r) \leq \mu(U(x,r)) \leq h(r)$ for every $x \in M$ and $r > 0$. It is shown in the fifth chapter that if a separable metric space admits an almost uniform measure, then the measure is uniquely determined in the framework of a certain equivalence. An analogous result is shown for uniform measures. The main result says that whenever a separable metric space admits an almost uniform measure then the comparison principle holds, and therefore the universal weak inequality-type differentiation theorem holds.

The last chapter contains examples. Some of them are taken from literature and modified: Davies's example of a compact metric space with two different finite Borel measures

agreeing on closed balls, Mattila's example of a compact metric space with uniform measure μ such that the differentiation theorem in convergence μ-almost everywhere does not hold. Further example of a compact metric space with two singular finite measures ν and μ such that $\lim_{r\to 0} \nu(U(x,r))/\mu(U(x,r)) = \infty$ μ-almost everywhere is based on an idea of D. Preiss and serves as a basis of a counterexample that the density theorem and the differentiation theorem for singular measures does not imply the integral differentiation theorem (in convergence μ-almost everywhere). Further example shows the existence of a compact metric space which does not admit a uniform measure but which admits an almost uniform measure. Moreover, quite simple sufficient condition for the existence of an isometric embedding into a compact metric space with a uniform measure is given. The last example shows that the universal inequality-type differentiation theorem does not imply the universal differentiation theorem in convergence in measure.