*Combinatorics, Probability & Computing* (1995) **4** 407–417

# CONDITIONAL INDEPENDENCES AMONG FOUR RANDOM VARIABLES II.[1]

F. MATÚŠ[2]

Numerous new properties of stochastic conditional independence are introduced. They are aimed, together with two surprisingly trivial examples, at a further reduction of the problem of probabilistic representability for four–element sets, i.e. of the problem which conditional independences within a system of four random variables can occur simultaneously. Proofs are based on fundamental properties of conditional independence and, in discrete case, on use of $I$–divergence and algebraic manipulations with marginal probabilities. A duality question is answered in negative.[3]

## 1. Introduction

Let $N = \{1,2,3,4\}$ and $\mathcal{S}$ be the family of all 56 couples $(ij|K)$, where $K \subset N$ and $ij$ is the union of two singletons $i$ and $j$ of $N - K$. Elements and singletons of $N$ are not distinguished and the sign for union between subsets of $N$ is omitted.

A relation $\mathcal{L} \subset \mathcal{S}$ is called *probabilistically (p–) representable* if there exists a system of (four) random variables $\xi = (\xi_i)_{i \in N}$ such that

$$\mathcal{L} = [\![\xi]\!] = \{(ij|K) \in \mathcal{S};\ \xi : i \perp j|K\}.$$

The expression $\xi : i \perp j|K$ shortens the phrase "$\xi_i$ is conditionally independent of $\xi_j$ given $\xi_K$", where $\xi_K = (\xi_k)_{k\in K}$, $K \subset N$, is a subsystem of $\xi$ ($\xi_\emptyset$ is taken as a constant). The system $\xi$, called *p–representation* of $\mathcal{L}$, is assumed to take only finite number of values. We reserve the symbol $\mathbf{P}$ for the class of all p–representable relations.

This paper presents further advances in the problem which relation $\mathcal{L} \subset \mathcal{S}$ is p–representable. To formulate them we have to describe a reduction of the problem attained in [10]. This task demands an extensive notation.

Let $\mathbf{S}_{12}^{(34|\emptyset)}$ denote the class of all relations $\mathcal{L} \subset \mathcal{S}$ which contain

$$\mathcal{M}_{12}^{(34|\emptyset)} = \{(34|1), (34|2), (12|\emptyset)\},$$

[1]This research was supported by Internal Grant 27564 of Czech Academy of Sciences.

[2]*Postal address:* Institute of Information Theory and Automation, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic. (matus@utia.cas.cz)

[3]*Key words and phrases:* Conditional independence, Probabilistic representations, Shannon entropy, $I$–divergence, Semimatroids, Matroids, Duality, Dependence/Independence structures/relations.

which are contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$, where

$$\mathcal{M}_{12} = \{(34|12), (12|\emptyset), (12|3), (12|4), (34|1), (34|2), (13|4), (23|4), (14|3), (24|3)\}$$

and

$$\mathcal{N}_{12} = \{(3|12), (4|12), (1|234), (2|134), (3|124), (4|123)\},$$

and which satisfy

$$(k|ij) \in \mathcal{L} \;\Leftrightarrow\; \{(kl|ij), (k|ijl)\} \subset \mathcal{L}\,, \quad i, j, k, l \in N \;\text{ distinct.}$$

Similarly, $\mathbf{S}_{12}^{(24|\emptyset)}$, $\mathbf{S}_{12}^{(13|2)}$, $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ will be the classes defined as above with the relations

$$\begin{aligned} \mathcal{M}_{12}^{(24|\emptyset)} &= \{(34|1), (24|3), (12|\emptyset)\}, & \mathcal{M}_{12}^{(13|2)} &= \{(12|3), (13|4), (34|2)\}, \\ \mathcal{M}_{12}^{(12|34)} &= \{(12|3), (12|4), (34|12)\}, & \mathcal{M}_{12}^{(13|24)} &= \{(12|3), (13|4), (34|12)\}, \end{aligned}$$

playing the role of $\mathcal{M}_{12}^{(34|\emptyset)}$, respectively. Every class $\mathbf{S}_{12}^{(\bullet)}$ has the cardinality $2^{11}$. Elements of the union

$$\mathbf{S}_{12} = \mathbf{S}_{12}^{(34|\emptyset)} \cup \mathbf{S}_{12}^{(24|\emptyset)} \cup \mathbf{S}_{12}^{(13|2)} \cup \mathbf{S}_{12}^{(12|34)} \cup \mathbf{S}_{12}^{(13|24)}$$

will be called *semimatroids*[4] here.

We have reduced (see [10]) the problem of p–representability for four-element sets to the question which semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ is p–representable. The following two theorems exhibit in a compressed form all results of the present paper and reduce the problem, roughly speaking, to the last less than $3 \cdot 2^5$ cases.

**Theorem 1.**

*A semimatroid $\mathcal{L}$ from $\mathbf{S}_{12}^{(34|\emptyset)}$ ($\mathbf{S}_{12}^{(12|34)}$) is p–representable if and only if*

$$\mathcal{L} \subset \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12} \qquad (\; \mathcal{L} \subset \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1), (34|2)\}\;)\,.$$

**Theorem 2.**

1. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(\bullet)} \quad \Rightarrow \quad \mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\,, \qquad (\bullet) \in \{(24|\emptyset), (13|2)\}\,,$
2. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)} \quad \Rightarrow \quad \mathcal{L} \subset \mathcal{M}_{12}^{(13|24)} \cup \{(12|\emptyset), (34|2), (24|3)\} \cup \{(2|134), (4|123)\}\,.$

Unlike in our previous paper, where all results on the problem were consequences of a few fundamental properties of Shannon entropies of subsystems and where the core was a convex analysis of special cones, proofs of these two theorems necessitate varied genuinely probabilistic arguments. Beside examples witnessing the p–representability of semimatroids from Theorem 1, a considerable number of subtle assertions of the type

$$(ij|K) \in [\![\xi]\!],\; \mathcal{M}_{12}^{(\bullet)} \subset [\![\xi]\!] \quad \Rightarrow \quad (\bullet) \in [\![\xi]\!]$$

will be demonstrated. Here $\xi$ is any system of four random variables, $(ij|K)$ a couple from $(\mathcal{M}_{12} \cup \mathcal{N}_{12}) - \mathcal{M}_{12}^{(\bullet)}$ forbidden in one of the theorems and $(\bullet)$ a couple admissible

[4]In the language of [10], the isomorphic images of all $\mathcal{L} \in \mathbf{S}_{12}$ exhaust the class of all semimatroids which are not Ingleton. For simplicity, we use here the word "semimatroid" in more narrow sense that in [10].

in the upper index position (which is always out of $\mathcal{M}_{12}$ and $\mathcal{N}_{12}$). Every assertion of this type entails that any p–representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ does not contain the very couple $(ij|K)$.

The proofs of both theorems result from a series of propositions organized as follows. First, we observe in Section 2 that p–representable semimatroids from $\mathbf{S}_{12}$ cannot contain a couple $(ij|k)$ along with the couple $(ik|j)$, $i, j, k$ distinct. This fact ensues from ten visually transparent schemes and the arguments behind them work even for four $\sigma$–algebras. Second, we deal with p–representable semimatroids from $\mathbf{S}_{12}^{(\bullet)}$, where $(\bullet)$ is one of the couples $(34|\emptyset)$, $(24|\emptyset)$ and $(13|2)$. As stated in Section 3 they can contain at most one couple $(34|12)$ from $\mathcal{M}_{12} - \mathcal{M}_{12}^{(\bullet)}$. Third, we continue with two classes $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ including functional dependence considerations (Section 4). A summary of the achieved arguments together with two examples of Section 5 will provide the desired two proofs. At the end a duality conjecture of [8] is refuted. With the exception of a few concluding remarks, this paper should be readable without any knowledge of the results and notations from [10].

## 2. The couples $(ij|k)$ and $(ik|j)$

Having four sub–$\sigma$–algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ in a probability space, the fundamental property of the conditional independence relation (see [1], [4], [5], [6], [12]) has the form

$$\mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_4 \quad \Leftrightarrow \quad \mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_3 \vee \mathcal{A}_4 \;\text{ and }\; \mathcal{A}_1 \perp \mathcal{A}_3 | \mathcal{A}_4 \ ,$$

where $\mathcal{A}_1 \vee \mathcal{A}_2$ is the smallest sub-$\sigma$-algebra containing $\mathcal{A}_1 \cup \mathcal{A}_2$. The projection technique of [11] and [14] provides also

$$\mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_3 \;\text{ and }\; \mathcal{A}_1 \perp \mathcal{A}_3 | \mathcal{A}_2 \quad \Leftrightarrow \quad \mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_2 \cap \mathcal{A}_3$$

and

$$\mathcal{A}_2 \perp \mathcal{A}_2 | \mathcal{A}_1 \quad \Leftrightarrow \quad \mathcal{A}_2 \subset \mathcal{A}_1$$

for complete $\sigma$–algebras (the completion of $\sigma$–algebras does not affect the conditional independences, cf. [10]). As a consequence of these three assertions we obtain the following result.

**Proposition 1.** *If a semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ contains two couples $(ij|k)$ and $(ik|j)$, where $i, j, k \in N$ are distinct, then it is not p–representable.*

**Proof.** To begin with an example of our argumentation, we shall show that the relation $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_\emptyset$ ($\mathcal{A}_\emptyset$ will be always the smallest sub–$\sigma$–algebra; this statement means that $\mathcal{A}_3$ is unconditionally independent of $\mathcal{A}_4$) ensues from

$$\mathcal{A}_3 \perp \mathcal{A}_1 | \mathcal{A}_4 \ , \quad \mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_1 \ , \quad \mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_2 \;\text{ and }\; \mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_\emptyset \ .$$

In virtue of this claim, any p–representable $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ cannot contain $(13|4)$ and, by permutations, the proof of the case $\mathbf{S}_{12}^{(34|\emptyset)} \subset \mathbf{S}_{12}$ will be over. In fact, first two independences imply $\mathcal{A}_3 \perp \mathcal{A}_1 \vee \mathcal{A}_4 | \mathcal{A}_1 \cap \mathcal{A}_4$ and thus $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_1 \cap \mathcal{A}_4$. Second two ones yield $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_3 | \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2 | \mathcal{A}_\emptyset$, respectively, which gives $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_\emptyset$ and then $\mathcal{A}_3 \perp \mathcal{A}_1 \cap \mathcal{A}_4 | \mathcal{A}_\emptyset$. Finally, we obtain $\mathcal{A}_3 \perp \mathcal{A}_4 \vee (\mathcal{A}_1 \cap \mathcal{A}_4) | \mathcal{A}_\emptyset$ meaning the desired $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_\emptyset$.

Before proceeding further we rewrite the above reasoning in a visually more tidy fashion.

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(3-1|4) \arrow[r] & (3-14|\underline{14}) \arrow[r] & (3-4|\underline{14}) \arrow[rr] & & (3-4|\emptyset) \\
(3-4|1) \arrow[ur] & & & & \\
(3-4|2) \arrow[r] & (\underline{14}-3|2) \arrow[r] & (\underline{14}-23|\emptyset) \arrow[r] & (3-\underline{14}|\emptyset) \arrow[uur] & \\
(1-2|\emptyset) \arrow[r] & (\underline{14}-2|\emptyset) \arrow[ur] & & &
\end{tikzcd}
\tag{1}
$$

The whole proof will consist of ten schemes of the above form.

We continue with the case $\mathcal{L} \in \mathbf{S}_{12}^{(24|\emptyset)}$ supposing $(34|2) \notin \mathcal{L}$ (otherwise $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$). Thus,

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(2-3|4) \arrow[r] & (2-34|\underline{34}) \arrow[r] & (2-4|\underline{34}) \arrow[r] & (2-4|\emptyset) \\
(2-4|3) \arrow[ur] & & & \\
(3-4|1) \arrow[r] & (\underline{34}-\underline{34}|1) \arrow[r] & (2-\underline{34}|\emptyset) \arrow[uur] & \\
(1-2|\emptyset) \arrow[urr] & & &
\end{tikzcd}
\tag{2}
$$

using $\mathcal{A}_3 \cap \mathcal{A}_4 \subset \mathcal{A}_1$ on the third and fourth line (this argument is employed only here and in (5)). The assertions (1) and (2) almost coincide with $(B1)$ and $(B2)$ of [13].[5] Next,

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(3-1|4) \arrow[r] & (4-3|\underline{14}) \arrow[r] & (4-2|\underline{14}) \arrow[r] & (2-4|\emptyset) \\
(3-4|1) \arrow[ur] & & & \\
(2-4|3) \arrow[r] & (4-2|3\underline{14}) \arrow[uur] & & \\
(1-2|\emptyset) \arrow[r] & (2-\underline{14}|\emptyset) \arrow[uuurr] & &
\end{tikzcd}
\tag{3}
$$

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(4-1|3) \arrow[r] & (4-3|\underline{13}) \arrow[r] & (4-2|\underline{13}) \arrow[r] & (2-4|\emptyset) \\
(4-3|1) \arrow[ur] & & & \\
(2-4|3) \arrow[r] & (4-2|3\underline{13}) \arrow[uur] & & \\
(1-2|\emptyset) \arrow[r] & (2-\underline{13}|\emptyset) \arrow[uuurr] & &
\end{tikzcd}
\tag{4}
$$

what closes the proof for $\mathbf{S}_{12}^{(24|\emptyset)}$ (the difference between (3) and (4) is only seemingly small; note $\underline{13} \leftrightarrow \underline{14}$).

To cover the case of semimatroids from $\mathbf{S}_{12}^{(13|2)}$ we need four very similar schemes

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(1-4|3) \arrow[r] & (1-3|\underline{34}) \arrow[r] & (1-23|\underline{34}) \arrow[r] & (1-3|2\underline{34}) \arrow[r] & (1-3|2) \\
(1-3|4) \arrow[ur] & & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{34}) \arrow[uur] & & & \\
(3-4|2) \arrow[r] & (\underline{34}-\underline{34}|2) \arrow[uuurrr] & & &
\end{tikzcd}
\tag{5}
$$

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
(3-4|1) \arrow[r] & (1-3|\underline{14}) \arrow[r] & (1-23|\underline{14}) \arrow[r] & (3-1|\underline{14}2) \arrow[r] & (1-3|2) \\
(3-1|4) \arrow[ur] & & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{14}) \arrow[uur] & & & \\
(3-4|2) \arrow[r] & (3-\underline{14}|2) \arrow[uuurrr] & & &
\end{tikzcd}
\tag{6}
$$

[5] I would like to thank to my colleague M. Studený for providing me manuscripts of his rather long proofs of $(B1), (B2)$ which settled the discrete setting.

$$
\begin{tikzcd}[column sep=small]
(3-2|4) \arrow[r] & (3-4|\underline{24}) \arrow[r] & (3-14|\underline{24}) \arrow[r] & (1-3|\underline{24}) \arrow[r] & (1-3|2) \\
(3-4|2) \arrow[ur] & & & & \\
(1-3|4) \arrow[r] & (3-1|4\underline{24}) \arrow[uur] & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{24}) \arrow[uuurrr] & & &
\end{tikzcd}
\tag{7}
$$

$$
\begin{tikzcd}[column sep=small]
(4-2|3) \arrow[r] & (3-4|\underline{23}) \arrow[r] & (3-14|\underline{23}) \arrow[r] & (1-3|\underline{23}) \arrow[r] & (1-3|2) \\
(4-3|2) \arrow[ur] & & & & \\
(1-3|4) \arrow[r] & (3-1|4\underline{23}) \arrow[uur] & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{23}) \arrow[uuurrr] & & &
\end{tikzcd}
\tag{8}
$$

Further, we concentrate on $\mathbf{S}_{12}^{(13|24)}$. If $(34|2) \in \mathcal{L}$ we have a former case as $\mathcal{L} \in \mathbf{S}_{12}^{(13|2)}$. On account of (5) and (6), respectively,

$$
\begin{tikzcd}[column sep=small]
(1-4|3) \arrow[dr] & & & \\
(1-3|4) \arrow[r] & (3-1|2\underline{34}) \arrow[r] & (3-14|2\underline{34}) \arrow[r] & (1-3|24) \\
(1-2|3) \arrow[ur] & & & \\
(3-4|12) \arrow[r] & (3-4|12\underline{34}) \arrow[uur] & &
\end{tikzcd}
\tag{9}
$$

$$
\begin{tikzcd}[column sep=small]
(3-4|1) \arrow[dr] & & & \\
(3-1|4) \arrow[r] & (3-1|2\underline{14}) \arrow[r] & (3-14|2\underline{14}) \arrow[r] & (1-3|24) \\
(1-2|3) \arrow[ur] & & & \\
(3-4|12) \arrow[r] & (3-4|12\underline{14}) \arrow[uur] & &
\end{tikzcd}
\tag{10}
$$

It remains to consider a p–representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ containing the two couples $(23|4)$ and $(24|3)$. However, permuting $1 \leftrightarrow 2$ in (9) we arrive at $(23|14) \in \mathcal{L}$, a contradiction with $\mathcal{L} \subset \mathcal{M}_{12} \cup \mathcal{N}_{12}$.

Finally, for $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ it suffices to exclude $(13|4) \in \mathcal{L}$ as then $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ and by permutations also $(23|4), (14|3), (24|3) \in \mathcal{L}$. ∎

## 3. Semimatroids from $\mathbf{S}_{12}^{(34|\emptyset)}$, $\mathbf{S}_{12}^{(24|\emptyset)}$ and $\mathbf{S}_{12}^{(13|2)}$

Though the results of the previous section concerned the most general situations, from now on we are not able to keep further this level and start to assume discrete systems $\xi = (\xi_i)_{i \in N}$. It is, however, a challenge to find respective proofs for $\sigma$–algebras.

The notation is fixed as follows. Every random variable $\xi_i$ takes its values in a finite set $X_i$, $i \in N$, and these sets are chosen to be minimal, i.e. every value $x_i \in X_i$ has a positive probability. The subsystems $\xi_I$ range over the Cartesian products $X_I = \Pi_{i \in I} X_i$, $I \subset N$ $(X_\emptyset = \{\emptyset\})$. Marginal probabilities like $\Pr(\xi_1 = x_1, \xi_3 = x_3)$ are shortened to $(x_1 x_3)$ and, occasionally, to $p_{13}(x_1, x_3)$.

**Proposition 2.** *If* $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ *is a p–representable semimatroid and* $(\bullet)$ *one of the couples* $(34|\emptyset), (24|\emptyset)$ *and* $(13|2)$ *then*

$$
\mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\ .
$$

**Proof.** To perform the demonstration for $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ is suffices (by Proposition 1) to establish

$$\{(12|3),(34|1),(34|2),(12|\emptyset)\} \subset [\![\xi]\!] \quad \Rightarrow \quad (34|\emptyset) \in [\![\xi]\!] \, .$$

The first three assumed conditional independences afford

$$(x_1x_3x_4)(x_2x_3x_4)(x_1)(x_2) = (x_1x_3)(x_1x_4)(x_2x_3)(x_2x_4) = (x_1x_2x_3)(x_3)(x_1x_4)(x_2x_4)$$

(we have been strictly sticking to the context notation $x_1 \in X_1$, $x_2 \in X_2$, etc.). This being so the nonnegative function $q_{34}$ defined on $X_{34}$ by

$$q_{34}(x_3,x_4) = \frac{(x_3x_4)^2}{(x_3)(x_4)} = \sum_{x_1\in X_1}\sum_{x_2\in X_2}\frac{(x_1x_2x_3)(x_1x_4)(x_2x_4)}{(x_1)(x_2)(x_4)}$$

is due to $\xi : 1 \perp 2|\emptyset$ a probability distribution

$$\sum_{x_4\in X_4}\sum_{x_3\in X_3} q_{34}(x_3,x_4) = \sum_{x_4\in X_4}(x_4) = 1 \ .$$

The marginal distribution $p_{34}$ of $\xi_3$ and $\xi_4$ is absolutely continuous w.r.t. the distribution $q_{34}$ and then the $I$–divergence (see [3])

$$I(p_{34}\|q_{34}) = \sum_{x_3\in X_3}\sum_{x_4\in X_4} p_{34}(x_3,x_4)\ln\frac{p_{34}(x_3,x_4)}{q_{34}(x_3,x_4)} = \sum_{x_3\in X_3}\sum_{x_4\in X_4}(x_3x_4)\ln\frac{(x_3)(x_4)}{(x_3x_4)}$$

is a finite nonnegative number vanishing if and only if $p_{34} = q_{34}$. Therefore

$$0 \le I(p_{34}\|q_{34}) = h_\xi(34) - h_\xi(3) - h_\xi(4) \, ,$$

where $h_\xi(I), I \subset N$, is the Shannon entropy of the subsystem $\xi_I$. The opposite inequality is, however, well-known (consider $I(p_{34}\|r_{34}) \ge 0$ with $r_{34}(x_3,x_4) = (x_3)(x_4)$). The equality entails $\xi : 3 \perp 4|\emptyset$.

Second, we have to prove that any p–representable semimatroid $\mathcal{L} = [\![\xi]\!] \in \mathbf{S}_{12}^{(24|\emptyset)}$ cannot contain the couples $(12|3)$ and $(12|4)$. Namely, this would cause $(24|\emptyset) \in \mathcal{L}$ as will be shown in the following two minisections.

If $(12|3) \in \mathcal{L}$ then

$$(x_1x_3x_4)(x_2x_3x_4)(x_1)(x_3) = (x_1x_3)(x_1x_4)(x_2x_3)(x_3x_4) = (x_1x_2x_3)(x_3)(x_1x_4)(x_3x_4)$$

from which

$$(x_3x_4)(x_2x_3x_4) = (x_3x_4)\sum_{x_1\in X_1}\frac{(x_1x_2x_3)(x_1x_4)}{(x_1)} \, .$$

For $(x_3x_4) = 0$ we have $(x_2x_3x_4) = 0$, $x_2 \in X_2$, and also

$$0 = (x_1x_3x_4)(x_1)(x_2x_3) = (x_1x_3)(x_1x_4)(x_2x_3) = (x_1x_2x_3)(x_3)(x_1x_4), \;\; x_1 \in X_1,$$

what implies that the sum equals zero. Anyway, after cancellation (if needed)

$$(x_2x_3x_4) = \sum_{x_1\in X_1}\frac{(x_1x_2x_3)(x_1x_4)}{(x_1)} \, .$$

The summation over $x_3 \in X_3$ together with $(12|\emptyset) \in \mathcal{L}$ yield $(24|\emptyset) \in \mathcal{L}$.

If $(12|4) \in \mathcal{L}$ then owing to $(34|1) \in \mathcal{L}$

$$(x_1x_2x_4)(x_4)(x_1x_3) = (x_1x_3x_4)(x_1)(x_2x_4)$$

and using also $(24|3) \in \mathcal{L}$ we obtain

$$(x_2x_3)(x_4) \sum_{x_1 \in X_1} \frac{(x_1x_2x_4)(x_1x_3)}{(x_1)} = (x_2x_3)(x_3x_4)(x_2x_4) = (x_2x_3x_4)(x_3)(x_2x_4)\,.$$

Thus,

$$(x_2x_3) \sum_{x_4 \in X_4} \sum_{x_1 \in X_1} \frac{(x_1x_2x_4)(x_1x_3)}{(x_1)} = (x_3) \sum_{x_4 \in X_4} \frac{(x_2x_3x_4)(x_2x_4)}{(x_4)}$$

and $(12|\emptyset) \in \mathcal{L}$ affords

$$(x_2x_3) = \sum_{x_4 \in X_4} \frac{(x_2x_3x_4)(x_2x_4)}{(x_2)(x_4)}\,.$$

On account of the last formula we see immediately that the function $q_{24}$ defined in the first part of the proof $(2 \leftrightarrow 3)$ is a probability distribution on $X_{24}$ and becose of it $(24|\emptyset) \in \mathcal{L}$ similarly as before.

Third, we step to the case $\mathcal{L} = [\![\xi]\!] \in \mathbf{S}_{12}^{(13|2)}$. If $(12|\emptyset) \in \mathcal{L}$ then the permutation $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ bring over $\mathcal{L}$ to the previous case and thus this incidence has been yet settled. It remains to prove $(12|4) \notin \mathcal{L}$ pursuant to Proposition 1. In fact, the opposite combined with $(13|4), (34|2) \in \mathcal{L}$ ensures

$$(x_1x_2x_4)(x_3x_4)(x_2x_3) = (x_1x_3x_4)(x_2x_4)(x_2x_3) = (x_1x_3x_4)(x_2x_3x_4)(x_2)$$

and then

$$(x_3x_4) \sum_{x_2 \in X_2} \frac{(x_1x_2x_4)(x_2x_3)}{(x_2)} = (x_3x_4)(x_1x_3x_4)\,.$$

Here we can formally cancel $(x_3x_4)$ as from $(x_3x_4) = 0$ one appoints $0 = (x_2x_3x_4)(x_2) = = (x_2x_3)(x_2x_4)$ and thus $0 = (x_2x_3)(x_1x_2x_4)$ for any $x_2 \in X_2$. But then the probability distribution

$$q_{123}(x_1, x_2, x_3) = \frac{(x_1x_2)(x_2x_3)}{(x_2)}$$

on $X_{123}$ has the marginal

$$\sum_{x_2 \in X_2} q_{123}(x_1, x_2, x_3) = (x_1x_3)\,.$$

Due to $(12|3) \in \mathcal{L}$ it is absolutely continuous w.r.t. the marginal distribution $p_{123}$ (note that $(x_1x_2x_3) = 0$ implies $0 = (x_1x_3)(x_2x_3) \geq (x_2x_3)q_{123}(x_1, x_2, x_3)$) and, moreover,

$$0 = h_\xi(13) + h_\xi(23) - h_\xi(123) - h_\xi(3) \leq \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \sum_{x_3 \in X_3} \frac{(x_1x_2)(x_2x_3)}{(x_2)} \ln \frac{(x_1x_2)(x_3)}{(x_1x_3)(x_2)}$$

(on the right-hand side is the $I$–divergence $I(q_{123} \| p_{123})$). This inequality can be casted to

$$h_\xi(23) + h_\xi(12) \leq h_\xi(123) + h_\xi(2)\,.$$

The semimodularity of $h_\xi$ forces the equality, whence $(13|2) \in \mathcal{L}$. ∎

## 4. Semimatroids from $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$

We continue with the notation of the previous section.

**Proposition 3.** *The intersection* $\mathcal{L} \cap \{(12|\emptyset),(13|4),(23|4),(14|3),(24|3)\}$ *is empty as soon as* $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ *is p–representable. If* $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ *then* $(23|4) \notin \mathcal{L}$.

**Proof.** Let $\xi$ be a p–representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(12|34)}$ and $(12|\emptyset) \in \mathcal{L}$. Then

$$(x_1x_2x_3x_4) = \frac{(x_1x_2x_3)(x_1x_2x_4)}{(x_1)(x_2)} = \frac{(x_1x_3)(x_2x_3)(x_1x_4)(x_2x_4)}{(x_1)(x_2)(x_3)(x_4)}$$

and it is easy to see

$$(x_1x_2x_3x_4)(y_1y_2x_3x_4) = (y_1x_2x_3x_4)(x_1y_2x_3x_4), \quad y_1 \in X_1, y_2 \in X_2,$$

i.e. $(12|34) \in \mathcal{L}$ (cf. also [6], the equivalent definition of the conditional independence by a factorization property). Hence, a semimatroid $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(12|34)}$ cannot contain $(12|\emptyset)$.

If $(13|4) \in \mathcal{L}$ we fix $x_4 \in X_4$ and denote by $Y_i = \{x_i \in X_i; (x_ix_4) > 0\}$, $i = 1,2,3$. Let us consider these two probability distributions on $Y_{123} = Y_1 \times Y_2 \times Y_3$

$$q(y_1,y_2,y_3) = \frac{(y_1y_2y_3x_4)}{(x_4)} \quad \text{and} \quad r(y_1,y_2,y_3) = \frac{(y_1x_4)(y_2y_3x_4)}{(x_4)^2} \ .$$

In other words (since $(12|4) \in \mathcal{L}$ the probability $(y_1y_2)$ is positive for $y_1 \in Y_1$ and $y_2 \in Y_2$)

$$q(y_1,y_2,y_3) = \frac{(y_1y_3)(y_2y_3)(y_1x_4)(y_2x_4)}{(y_1y_2)(y_3)(x_4)^2} \ .$$

The distribution $r$ is absolutely continuous w.r.t. $q$ as $(y_1y_3) > 0$ is valid due to $(13|4) \in \mathcal{L}$. It is, however, clearly the other way round what offers possibility to write

$$I(q\|r) + I(r\|q) = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \sum_{y_3 \in Y_3} [q(y_1,y_2,y_3) - r(y_1,y_2,y_3)] \ln \frac{(y_1y_3)(y_2y_3)(y_2x_4)}{(y_1y_2)(y_3)(y_2y_3x_4)} = 0 \ .$$

The point is that both $q$ and $r$ have the same all three two–dimensional marginals. Thus $q = r$ whence $(12|34) \in \mathcal{L}$ and we have proved the first assertion. (Another reasoning might us the observation that $q$ and $r$ are factorizable w.r.t. the marginals and then they are both maximum entropy extensions from them. Uniqueness of the extension brings $q = r$, see [2], [6].)

Let $\xi$ be a p–representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(13|24)}$ and $(23|4) \in \mathcal{L}$. We proceed as above with the distributions

$$q(y_1,y_2,y_3) = \frac{(y_1y_2y_3x_4)}{(x_4)} = \frac{(y_1y_3)(y_2y_3)(y_1y_2x_4)}{(y_1y_2)(y_3)(x_4)} \quad \text{and} \quad r(y_1,y_2,y_3) = \frac{(y_3x_4)(y_1y_2x_4)}{(x_4)^2} \ .$$

Note that if $(y_1y_2) = 0$ for some $y_1 \in Y_1$ and $y_2 \in Y_2$ then by $(12|3) \in \mathcal{L}$ also the expression $(y_1y_3)(y_2y_3)$ vanishes for any $y_3 \in Y_3$ and by $(13|4),(24|3) \in \mathcal{L}$ also $(y_1x_4)(y_3x_4)(y_2x_4) = 0$, a contradiction. The distributions $q$ and $r$ have the same supports ($(y_1y_3)(y_2y_3) > 0$ owing to $(13|4),(23|4) \in \mathcal{L}$) and along the same guidelines as before we conclude $q = r$; i.e. the desired $(13|24) \in \mathcal{L}$. ∎

Our last proposition concerns the functional dependence claims (cf. [7]). Note that e.g. $(1|23) \in [\![\xi]\!]$ means that the variable $\xi_1$ is a function of $\xi_{23}$ (in the $\sigma$–algebras environment $\mathcal{A}_1 \subset \mathcal{A}_2 \vee \mathcal{A}_3$, for $\mathcal{A}_2$ or $\mathcal{A}_3$ complete).

**Proposition 4.** *Every p–representable semimatroid from* $\mathbf{S}_{12}^{(12|34)}$ *has empty intersection with the class* $\mathcal{N}_{12} \subset \mathcal{S}$. *If* $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ *then the couples* $(1|234)$ *and* $(3|124)$ *are not elements of* $\mathcal{L}$.

**Proof.** Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(1|234)\}$. Suppose that $(x_1x_3x_4)(y_1x_3x_4) > 0$ and chose $x_2, y_2 \in X_2$ such that $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$. Then $(y_1x_3)(x_2x_3) > 0$ and by $(12|3) \in \mathcal{L}$ also $(y_1x_2x_3) > 0$; similarly $(y_1x_2x_4) > 0$ using $(12|4) \in \mathcal{L}$. The incidence $(34|12) \in \mathcal{L}$ brings thus $(y_1x_2x_3x_4) > 0$. Since $\xi_1$ is a function of $\xi_{234}$ the inequality $(x_1x_2x_3x_4)(y_1x_2x_3x_4) > 0$ implies $x_1 = y_1$. This means that $\xi_1$ is even a function of $\xi_{34}$ and consequently $(12|34) \in \mathcal{L}$ (cf. [11], [14]).

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(3|124)\}$. Then

$$(x_1x_2x_3x_4)(x_1x_2)\,(y_1y_2x_3x_4)(y_1y_2) = (y_1x_2x_3x_4)(y_1x_2)\,(x_1y_2x_3x_4)(x_1y_2)$$

and if $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$ we can drop out $x_3$ throughout this equation. Further casting leads to

$$(x_1x_4)(x_2x_4)(y_1x_4)(y_2x_4)\,[(x_1x_2)(y_1y_2) - (y_1x_2)(x_1y_2)] = 0$$

what affords

$$(x_1x_2x_3x_4)(y_1y_2x_3x_4) = (y_1x_2x_3x_4)(x_1y_2x_3x_4)\ .$$

However, if the left-hand side is equal to zero here then $(x_1x_3)(x_2x_3)(x_1x_4)(x_2x_4) = 0$ or $(y_1x_3)(y_1x_4)(y_2x_3)(y_2x_4) = 0$ and thus also the right-hand side vanishes. Consequently, $(12|34) \in \mathcal{L}$ and we finished the proof for $\mathbf{S}_{12}^{(12|34)}$.

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(1|234)\}$. We shall demonstrate that $\xi_1$ is a function of $\xi_{24}$ similarly as at the beginning. If $(x_1x_2x_4)(y_1x_2x_4) > 0$ then for some $x_3, y_3 \in X_3$ also $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$ . Owing to $(13|4) \in \mathcal{L}$ we realize $(x_1y_3x_4) > 0$ whence $(x_1y_3)(x_2y_3) > 0$. By $(12|3) \in \mathcal{L}$ we get $(x_1x_2y_3) > 0$ and knowing that $(x_1x_2x_4)$ is positive the incidence $(34|12) \in \mathcal{L}$ implies $(x_1x_2y_3x_4) > 0$. As $\xi_1$ is a function of $\xi_{234}$ and $(y_1x_2y_3x_4)(x_1x_2y_3x_4) > 0$ we arrive at $x_1 = y_1$, i.e. $\xi_1$ is a function of $\xi_{24}$, too.

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(3|124)\}$ and thus $(3|12) \in \mathcal{L}$. Starting with the assumptons $(x_2x_3x_4)(x_2y_3x_4) > 0$ and $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$ for properly chosen $x_1, y_1 \in X_1$, we infer $(x_1y_3x_4) > 0$ from $(13|4) \in \mathcal{L}$ and then $(x_1x_2y_3) > 0$ from $(12|3) \in \mathcal{L}$. Plainly, $x_3 = y_3$ what reads "$\xi_3$ is a function of $\xi_{24}$" and hence $(13|24) \in \mathcal{L}$. ∎

## 5. Examples and conclusions

Collecting assertions of Propositions 2, 3 and 4 we recognize immediately that Theorem 2 is their consequence. Also the implication $\Rightarrow$ of Theorem 1 follows from them. To prove the converse one we present two simple examples and refer the reader to Consequence 2 of [10] which designates that a semimatroid from $\mathbf{S}_{12}$ contained in a p–representable semimatroid from $\mathbf{S}_{12}$ is p–representable, too.

**Example 1.** Let $\Omega = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ be a four–element probability space with the uniform distribution and $\xi_1 = (\mathsf{ab})(\mathsf{cd})$, $\xi_2 = (\mathsf{ac})(\mathsf{bd})$, $\xi_3 = (\mathsf{a})(\mathsf{bcd})$ and $\xi_4 = (\mathsf{abc})(\mathsf{d})$ be four random variables specified as partitions. We claim

$$[\![\xi]\!] = \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\ .$$

To see it one has to verify the inclusion $\supset$ and $(34|\emptyset) \notin [\![\xi]\!]$. Then Theorem 2 of [10] forces $[\![\xi]\!]$ to be contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$ and the "only if" part of Theorem 1 entails the desired equality. ∎

**Example 2.** Let $\Omega$ have seven elements $\mathsf{a}, \mathsf{b}, \ldots, \mathsf{g}$ and be endowed with the uniform distribution. The system $\xi$ of four random variables $\xi_1 = (\mathsf{aceg})(\mathsf{bdf})$, $\xi_2 = (\mathsf{abef})(\mathsf{cdg})$, $\xi_3 = (\mathsf{abcdeg})(\mathsf{f})$ and $\xi_4 = (\mathsf{abcdef})(\mathsf{g})$ is a p–representation of the semimatroid

$$[\![\xi]\!] = \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1)(34|2)\}$$

(the argumentation is as before). ∎

In [8] we found an interesting symmetry on the family $\mathcal{R} = \{(ij|K) \in \mathcal{S}; i \neq j\}$, which was motivated by the matroid duality (see [8], [15])

$$(ij|K) \to (ij|K)^\neg = (ij|N - ijK)\ .$$

A natural open question was whether the dual $\mathcal{L}^\neg = \{(ij|K); (ij|K)^\neg \in \mathcal{L}\}$ of a $p$–representable relation $\mathcal{L} \subset \mathcal{R}$ is necessarily $p$–representable. In spite of some positive indications the answer is negative. A counterexample is provided by this semimatroid (Theorem 1)

$$\mathcal{L} = \mathcal{L}_{12}^{(34|\emptyset)} \cap \mathcal{R} = \{(34|1), (34|2), (12|\emptyset), (34|12)\} \in \mathbf{P}$$

with its dual

$$\mathcal{L}^\neg = \{(34|1), (34|2), (12|34), (34|\emptyset)\}$$

(permuting $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ we see from Theorem 1

$$\{(12|3), (12|4), (12|\emptyset), (34|12)\} \in \mathbf{S}_{12}^{(12|34)} - \mathbf{P}\ ).$$

The same question for matroids remains, however, open.

Speaking precisely in terms of [10] the elements of $\mathbf{S}_{12}$ are, up to permutations, the semimatroids which are not Ingleton. The above two examples show for the first time representatives of types that are irreducible in $\mathbf{P}$ and not Ingleton (cf. Consequence 4 of [10]). Hence, there are at least thirteen irreducible types in the lattice $\mathbf{P}$. The final aim in the problem of p–representability for four-elements sets is to find all these irreducible types.

## References


[1] CARNAL, E. (1980). Independence conditionnelle permutable. *Ann. Inst. Henri Poincaré* **16** 39–47.

[2] CSISZÁR, I. (1975). $I$–divergence geometry of probability distributions and minimization problems. *The Annals of Probability* **3** 146–158.

[3] CSISZÁR, I. and KÖRNER, J. (1981). *Information Theory (Coding Theorems for discrete memoryless systems).* Akadémiai Kiadó, Budapest.

[4] DAWID, A.P. (1979) Conditional independence in statistical theory (with discussion). *J. Roy. Statist. Soc.* B **41** 1–31.

[5] DÖHLER, R. (1980). Zur bedingten Unabhängigkeit zufälliger Ereignisse (On the conditional independence of random events). *Theory Probability Appl.* **25** 628–634.

[6] LAURITZEN, S.L. (1989). *Lectures on Contingency Tables.* University of Aalborg Press (3rd edn.), Aalborg.

[7] MATÚŠ, F. (1991). Abstract functional dependency structures. *Theoretical Computer Science* **81** 117–126.

[8] MATÚŠ, F. (1992). Ascending and descending conditional independence relations. *Transactions of the Eleventh Prague Conference on Information Theory, Statistical Decision Functions and Random Processes,* Academia, Prague, Vol. B, 181–200.

[9] MATÚŠ, F. (1994). Probabilistic conditional independence structures and matroid theory: background. *International Journal of General Systems* **22** 185–196.

[10] MATÚŠ, F. and STUDENÝ, M. (1994). Conditional independences among four random variables I. (to appear in *Combinatorics, Probability and Computation*)

[11] MOUCHART, M. and ROLIN, J.M. (1984). A note on conditional independence with statistical applications. *Statistica* **44** 557–584.

[12] PEARL, J. (1988). *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufman, San Meteo, California.

[13] STUDENÝ, M. (1992). Structural semigraphoids. *International Journal of General Systems* **22** 207–217.

[14] VAN PUTTEN, C. and VAN SCHUPPEN, J.H. (1985). Invariance properties of the conditional independence relation. *The Annals of Probability* **13** 934–945.

[15] WELSH, D.J.A. (1976). *Matroid Theory.* Academic Press, London.