

CONDITIONAL INDEPENDENCES AMONG FOUR RANDOM VARIABLES II.¹

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Numerous new properties of stochastic conditional independence are introduced. They are aimed, together with two surprisingly trivial examples, at a further reduction of the problem of probabilistic representability for four-element sets, i.e. of the problem which conditional independences within a system of four random variables can occur simultaneously. Proofs are based on fundamental properties of conditional independence and, in discrete case, on use of I -divergence and algebraic manipulations with marginal probabilities. A duality question is answered in negative.³

1. Introduction

Let $N = \{1, 2, 3, 4\}$ and \mathcal{S} be the family of all 56 couples $(ij|K)$, where $K \subset N$ and ij is the union of two singletons i and j of $N - K$. Elements and singletons of N are not distinguished and the sign for union between subsets of N is omitted.

A relation $\mathcal{L} \subset \mathcal{S}$ is called *probabilistically (p-) representable* if there exists a system of (four) random variables $\xi = (\xi_i)_{i \in N}$ such that

$$\mathcal{L} = [\xi] = \{(ij|K) \in \mathcal{S}; \xi : i \perp j|K\}.$$

The expression $\xi : i \perp j|K$ shortens the phrase “ ξ_i is conditionally independent of ξ_j given ξ_K ”, where $\xi_K = (\xi_k)_{k \in K}$, $K \subset N$, is a subsystem of ξ (ξ_\emptyset is taken as a constant). The system ξ , called *p-representation* of \mathcal{L} , is assumed to take only finite number of values. We reserve the symbol \mathbf{P} for the class of all p-representable relations.

This paper presents further advances in the problem which relation $\mathcal{L} \subset \mathcal{S}$ is p-representable. To formulate them we have to describe a reduction of the problem attained in [10]. This task demands an extensive notation.

Let $\mathbf{S}_{12}^{(34|\emptyset)}$ denote the class of all relations $\mathcal{L} \subset \mathcal{S}$ which contain

$$\mathcal{M}_{12}^{(34|\emptyset)} = \{(34|1), (34|2), (12|\emptyset)\},$$

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which are contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$, where

$$\mathcal{M}_{12} = \{(34|12), (12|\emptyset), (12|3), (12|4), (34|1), (34|2), (13|4), (23|4), (14|3), (24|3)\}$$

and

$$\mathcal{N}_{12} = \{(3|12), (4|12), (1|234), (2|134), (3|124), (4|123)\},$$

and which satisfy

$$(k|ij) \in \mathcal{L} \Leftrightarrow \{(kl|ij), (k|ijl)\} \subset \mathcal{L}, \quad i, j, k, l \in N \text{ distinct.}$$

Similarly, $\mathbf{S}_{12}^{(24|\emptyset)}$, $\mathbf{S}_{12}^{(13|2)}$, $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ will be the classes defined as above with the relations

$$\begin{aligned} \mathcal{M}_{12}^{(24|\emptyset)} &= \{(34|1), (24|3), (12|\emptyset)\}, & \mathcal{M}_{12}^{(13|2)} &= \{(12|3), (13|4), (34|2)\}, \\ \mathcal{M}_{12}^{(12|34)} &= \{(12|3), (12|4), (34|12)\}, & \mathcal{M}_{12}^{(13|24)} &= \{(12|3), (13|4), (34|12)\}, \end{aligned}$$

playing the role of $\mathcal{M}_{12}^{(34|\emptyset)}$, respectively. Every class $\mathbf{S}_{12}^{(\bullet)}$ has the cardinality 2^{11} . Elements of the union

$$\mathbf{S}_{12} = \mathbf{S}_{12}^{(34|\emptyset)} \cup \mathbf{S}_{12}^{(24|\emptyset)} \cup \mathbf{S}_{12}^{(13|2)} \cup \mathbf{S}_{12}^{(12|34)} \cup \mathbf{S}_{12}^{(13|24)}$$

will be called *semimatroids*⁴ here.

We have reduced (see [10]) the problem of p-representability for four-element sets to the question which semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ is p-representable. The following two theorems exhibit in a compressed form all results of the present paper and reduce the problem, roughly speaking, to the last less than $3 \cdot 2^5$ cases.

Theorem 1.

A semimatroid \mathcal{L} from $\mathbf{S}_{12}^{(34|\emptyset)}$ ($\mathbf{S}_{12}^{(12|34)}$) is p-representable if and only if

$$\mathcal{L} \subset \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12} \quad (\mathcal{L} \subset \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1), (34|2)\}).$$

Theorem 2.

1. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(\bullet)} \Rightarrow \mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12}, \quad (\bullet) \in \{(24|\emptyset), (13|2)\},$
2. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)} \Rightarrow \mathcal{L} \subset \mathcal{M}_{12}^{(13|24)} \cup \{(12|\emptyset), (34|2), (24|3)\} \cup \{(2|134), (4|123)\}.$

Unlike in our previous paper, where all results on the problem were consequences of a few fundamental properties of Shannon entropies of subsystems and where the core was a convex analysis of special cones, proofs of these two theorems necessitate varied genuinely probabilistic arguments. Beside examples witnessing the p-representability of semimatroids from Theorem 1, a considerable number of subtle assertions of the type

$$(ij|K) \in [\xi], \quad \mathcal{M}_{12}^{(\bullet)} \subset [\xi] \Rightarrow (\bullet) \in [\xi]$$

will be demonstrated. Here ξ is any system of four random variables, $(ij|K)$ a couple from $(\mathcal{M}_{12} \cup \mathcal{N}_{12}) - \mathcal{M}_{12}^{(\bullet)}$ forbidden in one of the theorems and (\bullet) a couple admissible

⁴In the language of [10], the isomorphic images of all $\mathcal{L} \in \mathbf{S}_{12}$ exhaust the class of all semimatroids which are not Ingleton. For simplicity, we use here the word “semimatroid” in more narrow sense that in [10].

in the upper index position (which is always out of \mathcal{M}_{12} and \mathcal{N}_{12}). Every assertion of this type entails that any p-representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ does not contain the very couple $(ij|K)$.

The proofs of both theorems result from a series of propositions organized as follows. First, we observe in Section 2 that p-representable semimatroids from \mathbf{S}_{12} cannot contain a couple $(ij|k)$ along with the couple $(ik|j)$, i, j, k distinct. This fact ensues from ten visually transparent schemes and the arguments behind them work even for four σ -algebras. Second, we deal with p-representable semimatroids from $\mathbf{S}_{12}^{(\bullet)}$, where (\bullet) is one of the couples $(34|\emptyset)$, $(24|\emptyset)$ and $(13|2)$. As stated in Section 3 they can contain at most one couple $(34|12)$ from $\mathcal{M}_{12} - \mathcal{M}_{12}^{(\bullet)}$. Third, we continue with two classes $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ including functional dependence considerations (Section 4). A summary of the achieved arguments together with two examples of Section 5 will provide the desired two proofs. At the end a duality conjecture of [8] is refuted. With the exception of a few concluding remarks, this paper should be readable without any knowledge of the results and notations from [10].

2. The couples $(ij|k)$ and $(ik|j)$

Having four sub- σ -algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 in a probability space, the fundamental property of the conditional independence relation (see [1], [4], [5], [6], [12]) has the form

$$\mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_4 \Leftrightarrow \mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_3 \vee \mathcal{A}_4 \text{ and } \mathcal{A}_1 \perp \mathcal{A}_3 | \mathcal{A}_4 ,$$

where $\mathcal{A}_1 \vee \mathcal{A}_2$ is the smallest sub- σ -algebra containing $\mathcal{A}_1 \cup \mathcal{A}_2$. The projection technique of [11] and [14] provides also

$$\mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_3 \text{ and } \mathcal{A}_1 \perp \mathcal{A}_3 | \mathcal{A}_2 \Leftrightarrow \mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_2 \cap \mathcal{A}_3$$

and

$$\mathcal{A}_2 \perp \mathcal{A}_1 | \mathcal{A}_1 \Leftrightarrow \mathcal{A}_2 \subset \mathcal{A}_1$$

for complete σ -algebras (the completion of σ -algebras does not affect the conditional independences, cf. [10]). As a consequence of these three assertions we obtain the following result.

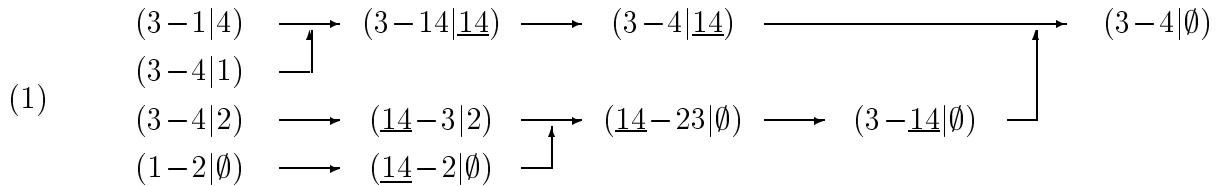
Proposition 1. *If a semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ contains two couples $(ij|k)$ and $(ik|j)$, where $i, j, k \in N$ are distinct, then it is not p-representable.*

Proof. To begin with an example of our argumentation, we shall show that the relation $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_\emptyset$ (\mathcal{A}_\emptyset will be always the smallest sub- σ -algebra; this statement means that \mathcal{A}_3 is unconditionally independent of \mathcal{A}_4) ensues from

$$\mathcal{A}_3 \perp \mathcal{A}_1 | \mathcal{A}_4 , \quad \mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_1 , \quad \mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_2 \text{ and } \mathcal{A}_1 \perp \mathcal{A}_2 | \mathcal{A}_\emptyset .$$

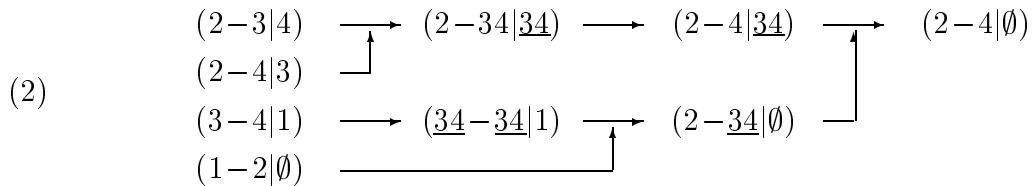
In virtue of this claim, any p-representable $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ cannot contain $(13|4)$ and, by permutations, the proof of the case $\mathbf{S}_{12}^{(34|\emptyset)} \subset \mathbf{S}_{12}$ will be over. In fact, first two independences imply $\mathcal{A}_3 \perp \mathcal{A}_1 \vee \mathcal{A}_4 | \mathcal{A}_1 \cap \mathcal{A}_4$ and thus $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_1 \cap \mathcal{A}_4$. Second two ones yield $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_3 | \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2 | \mathcal{A}_\emptyset$, respectively, which gives $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2 \vee \mathcal{A}_3 | \mathcal{A}_\emptyset$ and then $\mathcal{A}_3 \perp \mathcal{A}_1 \cap \mathcal{A}_4 | \mathcal{A}_\emptyset$. Finally, we obtain $\mathcal{A}_3 \perp \mathcal{A}_4 \vee (\mathcal{A}_1 \cap \mathcal{A}_4) | \mathcal{A}_\emptyset$ meaning the desired $\mathcal{A}_3 \perp \mathcal{A}_4 | \mathcal{A}_\emptyset$.

Before proceeding further we rewrite the above reasoning in a visually more tidy fashion.

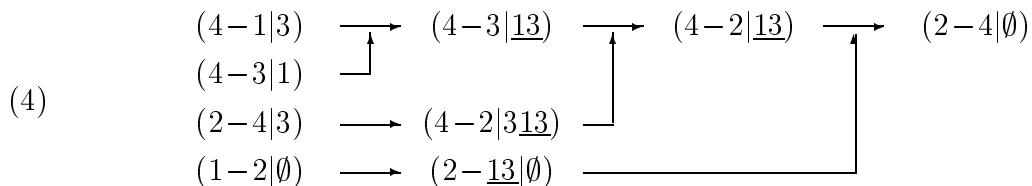
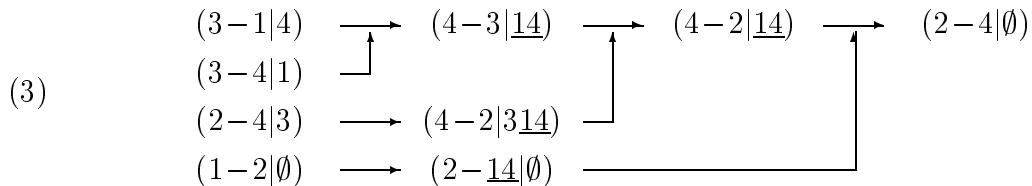


The whole proof will consist of ten schemes of the above form.

We continue with the case $\mathcal{L} \in \mathbf{S}_{12}^{(24|\emptyset)}$ supposing $(34|2) \notin \mathcal{L}$ (otherwise $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$). Thus,

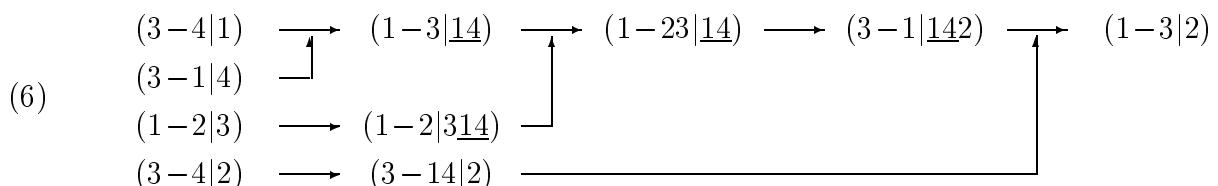
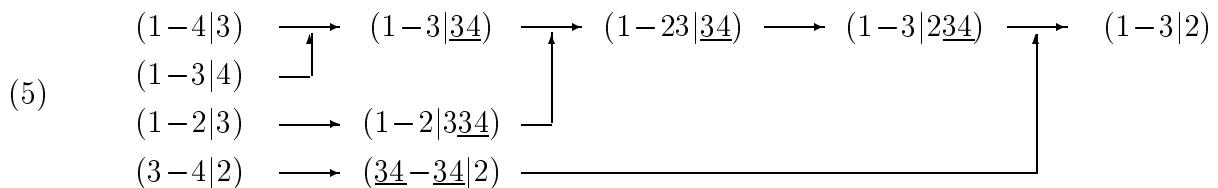


using $\mathcal{A}_3 \cap \mathcal{A}_4 \subset \mathcal{A}_1$ on the third and fourth line (this argument is employed only here and in (5)). The assertions (1) and (2) almost coincide with (B1) and (B2) of [13].⁵ Next,

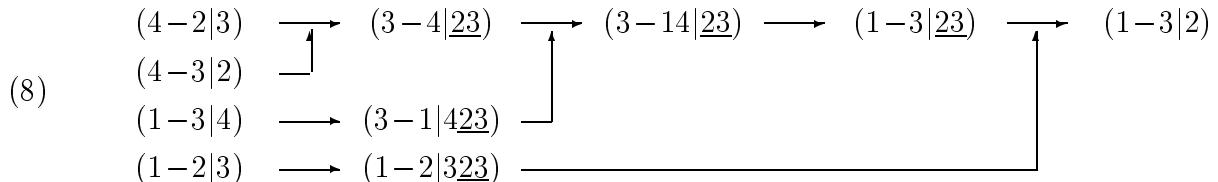
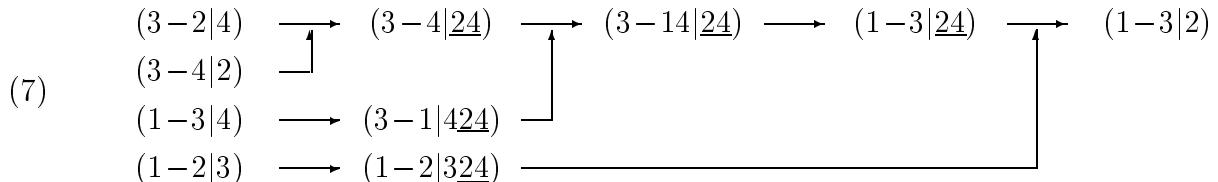


what closes the proof for $\mathbf{S}_{12}^{(24|\emptyset)}$ (the difference between (3) and (4) is only seemingly small; note $\underline{13} \leftrightarrow \underline{14}$).

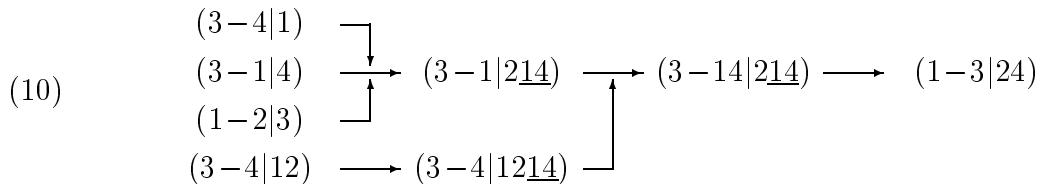
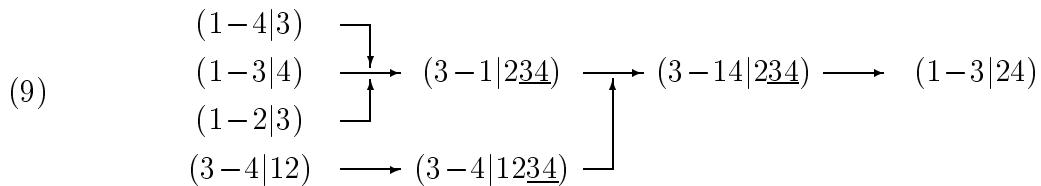
To cover the case of semimatroids from $\mathbf{S}_{12}^{(13|2)}$ we need four very similar schemes



⁵I would like to thank to my colleague M. Studený for providing me manuscripts of his rather long proofs of (B1), (B2) which settled the discrete setting.



Further, we concentrate on $\mathbf{S}_{12}^{(13|24)}$. If $(34|2) \in \mathcal{L}$ we have a former case as $\mathcal{L} \in \mathbf{S}_{12}^{(13|2)}$. On account of (5) and (6), respectively,



It remains to consider a p-representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ containing the two couples $(23|4)$ and $(24|3)$. However, permuting $1 \leftrightarrow 2$ in (9) we arrive at $(23|14) \in \mathcal{L}$, a contradiction with $\mathcal{L} \subset \mathcal{M}_{12} \cup \mathcal{N}_{12}$.

Finally, for $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ it suffices to exclude $(13|4) \in \mathcal{L}$ as then $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ and by permutations also $(23|4), (14|3), (24|3) \in \mathcal{L}$. \blacksquare

3. Semimatroids from $\mathbf{S}_{12}^{(34|\emptyset)}, \mathbf{S}_{12}^{(24|\emptyset)}$ and $\mathbf{S}_{12}^{(13|2)}$

Though the results of the previous section concerned the most general situations, from now on we are not able to keep further this level and start to assume discrete systems $\xi = (\xi_i)_{i \in N}$. It is, however, a challenge to find respective proofs for σ -algebras.

The notation is fixed as follows. Every random variable ξ_i takes its values in a finite set X_i , $i \in N$, and these sets are chosen to be minimal, i.e. every value $x_i \in X_i$ has a positive probability. The subsystems ξ_I range over the Cartesian products $X_I = \prod_{i \in I} X_i$, $I \subset N$ ($X_\emptyset = \{\emptyset\}$). Marginal probabilities like $\Pr(\xi_1 = x_1, \xi_3 = x_3)$ are shortened to $(x_1 x_3)$ and, occasionally, to $p_{13}(x_1, x_3)$.

Proposition 2. *If $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ is a p-representable semimatroid and (\bullet) one of the couples $(34|\emptyset), (24|\emptyset)$ and $(13|2)$ then*

$$\mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12} .$$

Proof. To perform the demonstration for $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ it suffices (by Proposition 1) to establish

$$\{(12|3), (34|1), (34|2), (12|\emptyset)\} \subset [\xi] \Rightarrow (34|\emptyset) \in [\xi].$$

The first three assumed conditional independences afford

$$(x_1 x_3 x_4)(x_2 x_3 x_4)(x_1)(x_2) = (x_1 x_3)(x_1 x_4)(x_2 x_3)(x_2 x_4) = (x_1 x_2 x_3)(x_3)(x_1 x_4)(x_2 x_4)$$

(we have been strictly sticking to the context notation $x_1 \in X_1$, $x_2 \in X_2$, etc.). This being so the nonnegative function q_{34} defined on X_{34} by

$$q_{34}(x_3, x_4) = \frac{(x_3 x_4)^2}{(x_3)(x_4)} = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \frac{(x_1 x_2 x_3)(x_1 x_4)(x_2 x_4)}{(x_1)(x_2)(x_4)}$$

is due to $\xi : 1 \perp 2|\emptyset$ a probability distribution

$$\sum_{x_4 \in X_4} \sum_{x_3 \in X_3} q_{34}(x_3, x_4) = \sum_{x_4 \in X_4} (x_4) = 1.$$

The marginal distribution p_{34} of ξ_3 and ξ_4 is absolutely continuous w.r.t. the distribution q_{34} and then the I -divergence (see [3])

$$I(p_{34} \| q_{34}) = \sum_{x_3 \in X_3} \sum_{x_4 \in X_4} p_{34}(x_3, x_4) \ln \frac{p_{34}(x_3, x_4)}{q_{34}(x_3, x_4)} = \sum_{x_3 \in X_3} \sum_{x_4 \in X_4} (x_3 x_4) \ln \frac{(x_3)(x_4)}{(x_3 x_4)}$$

is a finite nonnegative number vanishing if and only if $p_{34} = q_{34}$. Therefore

$$0 \leq I(p_{34} \| q_{34}) = h_\xi(34) - h_\xi(3) - h_\xi(4),$$

where $h_\xi(I)$, $I \subset N$, is the Shannon entropy of the subsystem ξ_I . The opposite inequality is, however, well-known (consider $I(p_{34} \| r_{34}) \geq 0$ with $r_{34}(x_3, x_4) = (x_3)(x_4)$). The equality entails $\xi : 3 \perp 4|\emptyset$.

Second, we have to prove that any p-representable semimatroid $\mathcal{L} = [\xi] \in \mathbf{S}_{12}^{(24|\emptyset)}$ cannot contain the couples $(12|3)$ and $(12|4)$. Namely, this would cause $(24|\emptyset) \in \mathcal{L}$ as will be shown in the following two minisections.

If $(12|3) \in \mathcal{L}$ then

$$(x_1 x_3 x_4)(x_2 x_3 x_4)(x_1)(x_3) = (x_1 x_3)(x_1 x_4)(x_2 x_3)(x_3 x_4) = (x_1 x_2 x_3)(x_3)(x_1 x_4)(x_3 x_4)$$

from which

$$(x_3 x_4)(x_2 x_3 x_4) = (x_3 x_4) \sum_{x_1 \in X_1} \frac{(x_1 x_2 x_3)(x_1 x_4)}{(x_1)}.$$

For $(x_3 x_4) = 0$ we have $(x_2 x_3 x_4) = 0$, $x_2 \in X_2$, and also

$$0 = (x_1 x_3 x_4)(x_1)(x_2 x_3) = (x_1 x_3)(x_1 x_4)(x_2 x_3) = (x_1 x_2 x_3)(x_3)(x_1 x_4), \quad x_1 \in X_1,$$

what implies that the sum equals zero. Anyway, after cancellation (if needed)

$$(x_2 x_3 x_4) = \sum_{x_1 \in X_1} \frac{(x_1 x_2 x_3)(x_1 x_4)}{(x_1)}.$$

The summation over $x_3 \in X_3$ together with $(12|\emptyset) \in \mathcal{L}$ yield $(24|\emptyset) \in \mathcal{L}$.

If $(12|4) \in \mathcal{L}$ then owing to $(34|1) \in \mathcal{L}$

$$(x_1 x_2 x_4)(x_4)(x_1 x_3) = (x_1 x_3 x_4)(x_1)(x_2 x_4)$$

and using also $(24|3) \in \mathcal{L}$ we obtain

$$(x_2 x_3)(x_4) \sum_{x_1 \in X_1} \frac{(x_1 x_2 x_4)(x_1 x_3)}{(x_1)} = (x_2 x_3)(x_3 x_4)(x_2 x_4) = (x_2 x_3 x_4)(x_3)(x_2 x_4).$$

Thus,

$$(x_2 x_3) \sum_{x_4 \in X_4} \sum_{x_1 \in X_1} \frac{(x_1 x_2 x_4)(x_1 x_3)}{(x_1)} = (x_3) \sum_{x_4 \in X_4} \frac{(x_2 x_3 x_4)(x_2 x_4)}{(x_4)}$$

and $(12|\emptyset) \in \mathcal{L}$ affords

$$(x_2 x_3) = \sum_{x_4 \in X_4} \frac{(x_2 x_3 x_4)(x_2 x_4)}{(x_2)(x_4)}.$$

On account of the last formula we see immediately that the function q_{24} defined in the first part of the proof $(2 \leftrightarrow 3)$ is a probability distribution on X_{24} and because of it $(24|\emptyset) \in \mathcal{L}$ similarly as before.

Third, we step to the case $\mathcal{L} = [\xi] \in \mathbf{S}_{12}^{(13|2)}$. If $(12|\emptyset) \in \mathcal{L}$ then the permutation $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ bring over \mathcal{L} to the previous case and thus this incidence has been yet settled. It remains to prove $(12|4) \notin \mathcal{L}$ pursuant to Proposition 1. In fact, the opposite combined with $(13|4), (34|2) \in \mathcal{L}$ ensures

$$(x_1 x_2 x_4)(x_3 x_4)(x_2 x_3) = (x_1 x_3 x_4)(x_2 x_4)(x_2 x_3) = (x_1 x_3 x_4)(x_2 x_3 x_4)(x_2)$$

and then

$$(x_3 x_4) \sum_{x_2 \in X_2} \frac{(x_1 x_2 x_4)(x_2 x_3)}{(x_2)} = (x_3 x_4)(x_1 x_3 x_4).$$

Here we can formally cancel $(x_3 x_4)$ as from $(x_3 x_4) = 0$ one appoints $0 = (x_2 x_3 x_4)(x_2) = (x_2 x_3)(x_2 x_4)$ and thus $0 = (x_2 x_3)(x_1 x_2 x_4)$ for any $x_2 \in X_2$. But then the probability distribution

$$q_{123}(x_1, x_2, x_3) = \frac{(x_1 x_2)(x_2 x_3)}{(x_2)}$$

on X_{123} has the marginal

$$\sum_{x_2 \in X_2} q_{123}(x_1, x_2, x_3) = (x_1 x_3).$$

Due to $(12|3) \in \mathcal{L}$ it is absolutely continuous w.r.t. the marginal distribution p_{123} (note that $(x_1 x_2 x_3) = 0$ implies $0 = (x_1 x_3)(x_2 x_3) \geq (x_2 x_3)q_{123}(x_1, x_2, x_3)$) and, moreover,

$$0 = h_\xi(13) + h_\xi(23) - h_\xi(123) - h_\xi(3) \leq \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \sum_{x_3 \in X_3} \frac{(x_1 x_2)(x_2 x_3)}{(x_2)} \ln \frac{(x_1 x_2)(x_3)}{(x_1 x_3)(x_2)}$$

(on the right-hand side is the I -divergence $I(q_{123} \| p_{123})$). This inequality can be casted to

$$h_\xi(23) + h_\xi(12) \leq h_\xi(123) + h_\xi(2).$$

The semimodularity of h_ξ forces the equality, whence $(13|2) \in \mathcal{L}$. ■

4. Semimatroids from $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$

We continue with the notation of the previous section.

Proposition 3. *The intersection $\mathcal{L} \cap \{(12|\emptyset), (13|4), (23|4), (14|3), (24|3)\}$ is empty as soon as $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ is p-representable. If $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ then $(23|4) \notin \mathcal{L}$.*

Proof. Let ξ be a p-representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(12|34)}$ and $(12|\emptyset) \in \mathcal{L}$. Then

$$(x_1 x_2 x_3 x_4) = \frac{(x_1 x_2 x_3)(x_1 x_2 x_4)}{(x_1)(x_2)} = \frac{(x_1 x_3)(x_2 x_3)(x_1 x_4)(x_2 x_4)}{(x_1)(x_2)(x_3)(x_4)}$$

and it is easy to see

$$(x_1 x_2 x_3 x_4)(y_1 y_2 x_3 x_4) = (y_1 x_2 x_3 x_4)(x_1 y_2 x_3 x_4), \quad y_1 \in X_1, y_2 \in X_2,$$

i.e. $(12|34) \in \mathcal{L}$ (cf. also [6], the equivalent definition of the conditional independence by a factorization property). Hence, a semimatroid $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(12|34)}$ cannot contain $(12|\emptyset)$.

If $(13|4) \in \mathcal{L}$ we fix $x_4 \in X_4$ and denote by $Y_i = \{x_i \in X_i; (x_i x_4) > 0\}$, $i = 1, 2, 3$. Let us consider these two probability distributions on $Y_{123} = Y_1 \times Y_2 \times Y_3$

$$q(y_1, y_2, y_3) = \frac{(y_1 y_2 y_3 x_4)}{(x_4)} \quad \text{and} \quad r(y_1, y_2, y_3) = \frac{(y_1 x_4)(y_2 y_3 x_4)}{(x_4)^2}.$$

In other words (since $(12|4) \in \mathcal{L}$ the probability $(y_1 y_2)$ is positive for $y_1 \in Y_1$ and $y_2 \in Y_2$)

$$q(y_1, y_2, y_3) = \frac{(y_1 y_3)(y_2 y_3)(y_1 x_4)(y_2 x_4)}{(y_1 y_2)(y_3)(x_4)^2}.$$

The distribution r is absolutely continuous w.r.t. q as $(y_1 y_3) > 0$ is valid due to $(13|4) \in \mathcal{L}$. It is, however, clearly the other way round what offers possibility to write

$$I(q\|r) + I(r\|q) = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \sum_{y_3 \in Y_3} [q(y_1, y_2, y_3) - r(y_1, y_2, y_3)] \ln \frac{(y_1 y_3)(y_2 y_3)(y_2 x_4)}{(y_1 y_2)(y_3)(y_2 y_3 x_4)} = 0.$$

The point is that both q and r have the same all three two-dimensional marginals. Thus $q = r$ whence $(12|34) \in \mathcal{L}$ and we have proved the first assertion. (Another reasoning might us the observation that q and r are factorizable w.r.t. the marginals and then they are both maximum entropy extensions from them. Uniqueness of the extension brings $q = r$, see [2], [6].)

Let ξ be a p-representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(13|24)}$ and $(23|4) \in \mathcal{L}$. We proceed as above with the distributions

$$q(y_1, y_2, y_3) = \frac{(y_1 y_2 y_3 x_4)}{(x_4)} = \frac{(y_1 y_3)(y_2 y_3)(y_1 y_2 x_4)}{(y_1 y_2)(y_3)(x_4)} \quad \text{and} \quad r(y_1, y_2, y_3) = \frac{(y_3 x_4)(y_1 y_2 x_4)}{(x_4)^2}.$$

Note that if $(y_1 y_2) = 0$ for some $y_1 \in Y_1$ and $y_2 \in Y_2$ then by $(12|3) \in \mathcal{L}$ also the expression $(y_1 y_3)(y_2 y_3)$ vanishes for any $y_3 \in Y_3$ and by $(13|4), (24|3) \in \mathcal{L}$ also $(y_1 x_4)(y_3 x_4)(y_2 x_4) = 0$, a contradiction. The distributions q and r have the same supports $((y_1 y_3)(y_2 y_3) > 0)$ owing to $(13|4), (23|4) \in \mathcal{L}$ and along the same guidelines as before we conclude $q = r$; i.e. the desired $(13|24) \in \mathcal{L}$. ■

Our last proposition concerns the functional dependence claims (cf. [7]). Note that e.g. $(1|23) \in [\xi]$ means that the variable ξ_1 is a function of ξ_{23} (in the σ -algebras environment $\mathcal{A}_1 \subset \mathcal{A}_2 \vee \mathcal{A}_3$, for \mathcal{A}_2 or \mathcal{A}_3 complete).

Proposition 4. *Every p-representable semimatroid from $\mathbf{S}_{12}^{(12|34)}$ has empty intersection with the class $\mathcal{N}_{12} \subset \mathcal{S}$. If $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ then the couples $(1|234)$ and $(3|124)$ are not elements of \mathcal{L} .*

Proof. Let $\mathcal{L} = [\xi] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(1|234)\}$. Suppose that $(x_1x_3x_4)(y_1x_3x_4) > 0$ and chose $x_2, y_2 \in X_2$ such that $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$. Then $(y_1x_3)(x_2x_3) > 0$ and by $(12|3) \in \mathcal{L}$ also $(y_1x_2x_3) > 0$; similarly $(y_1x_2x_4) > 0$ using $(12|4) \in \mathcal{L}$. The incidence $(34|12) \in \mathcal{L}$ brings thus $(y_1x_2x_3x_4) > 0$. Since ξ_1 is a function of ξ_{234} the inequality $(x_1x_2x_3x_4)(y_1x_2x_3x_4) > 0$ implies $x_1 = y_1$. This means that ξ_1 is even a function of ξ_{34} and consequently $(12|34) \in \mathcal{L}$ (cf. [11], [14]).

Let $\mathcal{L} = [\xi] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(3|124)\}$. Then

$$(x_1x_2x_3x_4)(x_1x_2)(y_1y_2x_3x_4)(y_1y_2) = (y_1x_2x_3x_4)(y_1x_2)(x_1y_2x_3x_4)(x_1y_2)$$

and if $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$ we can drop out x_3 throughout this equation. Further casting leads to

$$(x_1x_4)(x_2x_4)(y_1x_4)(y_2x_4) [(x_1x_2)(y_1y_2) - (y_1x_2)(x_1y_2)] = 0$$

what affords

$$(x_1x_2x_3x_4)(y_1y_2x_3x_4) = (y_1x_2x_3x_4)(x_1y_2x_3x_4) .$$

However, if the left-hand side is equal to zero here then $(x_1x_3)(x_2x_3)(x_1x_4)(x_2x_4) = 0$ or $(y_1x_3)(y_1x_4)(y_2x_3)(y_2x_4) = 0$ and thus also the right-hand side vanishes. Consequently, $(12|34) \in \mathcal{L}$ and we finished the proof for $\mathbf{S}_{12}^{(12|34)}$.

Let $\mathcal{L} = [\xi] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(1|234)\}$. We shall demonstrate that ξ_1 is a function of ξ_{24} similarly as at the beginning. If $(x_1x_2x_4)(y_1x_2x_4) > 0$ then for some $x_3, y_3 \in X_3$ also $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$. Owing to $(13|4) \in \mathcal{L}$ we realize $(x_1y_3x_4) > 0$ whence $(x_1y_3)(x_2y_3) > 0$. By $(12|3) \in \mathcal{L}$ we get $(x_1x_2y_3) > 0$ and knowing that $(x_1x_2x_4)$ is positive the incidence $(34|12) \in \mathcal{L}$ implies $(x_1x_2y_3x_4) > 0$. As ξ_1 is a function of ξ_{234} and $(y_1x_2y_3x_4)(x_1x_2y_3x_4) > 0$ we arrive at $x_1 = y_1$, i.e. ξ_1 is a function of ξ_{24} , too.

Let $\mathcal{L} = [\xi] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(3|124)\}$ and thus $(3|12) \in \mathcal{L}$. Starting with the assumptions $(x_2x_3x_4)(x_2y_3x_4) > 0$ and $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$ for properly chosen $x_1, y_1 \in X_1$, we infer $(x_1y_3x_4) > 0$ from $(13|4) \in \mathcal{L}$ and then $(x_1x_2y_3) > 0$ from $(12|3) \in \mathcal{L}$. Plainly, $x_3 = y_3$ what reads “ ξ_3 is a function of ξ_{24} ” and hence $(13|24) \in \mathcal{L}$. ■

5. Examples and conclusions

Collecting assertions of Propositions 2, 3 and 4 we recognize immediately that Theorem 2 is their consequence. Also the implication \Rightarrow of Theorem 1 follows from them. To prove the converse one we present two simple examples and refer the reader to Consequence 2 of [10] which designates that a semimatroid from \mathbf{S}_{12} contained in a p-representable semimatroid from \mathbf{S}_{12} is p-representable, too.

Example 1. Let $\Omega = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ be a four-element probability space with the uniform distribution and $\xi_1 = (\mathbf{ab})(\mathbf{cd})$, $\xi_2 = (\mathbf{ac})(\mathbf{bd})$, $\xi_3 = (\mathbf{a})(\mathbf{bcd})$ and $\xi_4 = (\mathbf{abc})(\mathbf{d})$ be four random variables specified as partitions. We claim

$$[\xi] = \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12} .$$

To see it one has to verify the inclusion \supset and $(34|\emptyset) \notin \llbracket \xi \rrbracket$. Then Theorem 2 of [10] forces $\llbracket \xi \rrbracket$ to be contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$ and the “only if” part of Theorem 1 entails the desired equality. ■

Example 2. Let Ω have seven elements a, b, \dots, g and be endowed with the uniform distribution. The system ξ of four random variables $\xi_1 = (aceg)(bdf)$, $\xi_2 = (abef)(cdg)$, $\xi_3 = (abcdeg)(f)$ and $\xi_4 = (abcdef)(g)$ is a p–representation of the semimatroid

$$\llbracket \xi \rrbracket = \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1)(34|2)\}$$

(the argumentation is as before). ■

In [8] we found an interesting symmetry on the family $\mathcal{R} = \{(ij|K) \in \mathcal{S}; i \neq j\}$, which was motivated by the matroid duality (see [8], [15])

$$(ij|K) \rightarrow (ij|K)^\neg = (ij|N - ijK).$$

A natural open question was whether the dual $\mathcal{L}^\neg = \{(ij|K); (ij|K)^\neg \in \mathcal{L}\}$ of a p–representable relation $\mathcal{L} \subset \mathcal{R}$ is necessarily p–representable. In spite of some positive indications the answer is negative. A counterexample is provided by this semimatroid (Theorem 1)

$$\mathcal{L} = \mathcal{L}_{12}^{(34|\emptyset)} \cap \mathcal{R} = \{(34|1), (34|2), (12|\emptyset), (34|12)\} \in \mathbf{P}$$

with its dual

$$\mathcal{L}^\neg = \{(34|1), (34|2), (12|34), (34|\emptyset)\}$$

(permuting $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ we see from Theorem 1

$$\{(12|3), (12|4), (12|\emptyset), (34|12)\} \in \mathbf{S}_{12}^{(12|34)} - \mathbf{P}.$$

The same question for matroids remains, however, open.

Speaking precisely in terms of [10] the elements of \mathbf{S}_{12} are, up to permutations, the semimatroids which are not Ingleton. The above two examples show for the first time representatives of types that are irreducible in \mathbf{P} and not Ingleton (cf. Consequence 4 of [10]). Hence, there are at least thirteen irreducible types in the lattice \mathbf{P} . The final aim in the problem of p–representability for four-elements sets is to find all these irreducible types.

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