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Representation of irrelevance relations by annotated graphs*

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Abstract. Irrelevance relations are sets of statements of the form: given that the 'value' of Z is known, the 'values' of Y can add no further information about the 'values' of X. Undirected Graphs (UGs), Directed Acyclic Graphs (DAGs) and Chain Graphs (CGs) were used and investigated as schemes for the purpose of representing irrelevance relations. It is known that, although all three schemes can approximate irrelevance, they are inadequate in the sense that there are relations which cannot be fully represented by anyone of them.

In this paper annotated graphs are defined and suggested as a new model for graphical representation. It is shown that this new model is a proper generalization of the former

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models: any irrelevance relation that can be represented by either one of the previous models can also be represented by an annotated graph, and there are relations that can be represented by an annotated graph but cannot be represented by either one of the former models. The question of whether this new model is powerful enough to represent all the irrelevance relations, as well as some other related questions, is still open.

Keywords: irrelevance relation, graphoid, (regular) annotated graph, membership algorithm, annotation algorithm

1. Introduction

1.1. Motivation

Any system that reasons about knowledge and beliefs must make use of information about dependencies and relevancies. If we have acquired a body of knowledge Z and wish to assess the truth value of a proposition X, it is important to know whether it would be worthwhile to consult another proposition Y, which is not in Z. In other words, before we examine Y, we need to know if its value can potentially generate new information relative to X, information not available from Z.

Many AI systems approach this problem in ad-hoc ways. These systems, though computationally convenient, are semantically sloppy. They often yield surprising and counterintuitive conclusions - see [5].

The other approach to the problem of dealing with irrelevance, as with any other notion involving uncertainty, is to handle it within probability theory, which is an appropriate mathematical framework. The problem with this approach is that it cures the problem of lack of semantics, but introduces computational inefficiency.

The goal of the theory of *graphoids* is to make probabilistic systems operational by making relevance relationships explicit. The theory developed may have some applications to relational databases too. The representation has to be made in a way which will make it easy to identify the facts which are irrelevant and therefore can be neglected, or, even better, make it easy to identify the relevant facts, which must be considered.

The paper introduces new ways of storing the information included in irrelevance relations in graphs, via a semantical interpretation of graph's cutsets - to be described in the text. While the size of irrelevance relations is usually exponential in the number of variables involved, the graphs themselves are polynomial constructs.

1.2. Organization

The rest of the paper is organized as follows. In the next section we present the background needed for our discussion. The original results of this paper appear in Section 3. Sections 4, 5 and 7 contain technical proofs and an example concerning our annotation algorithm. In Section

6.1 we show that our results are a proper improvement upon the previous related works. In Section 6.2 we state some open problems which arise from these new results.

2. Background

As the subject is relatively new (but developing fast) we will try to make our presentation self contained. This section is an introductory section providing the basic definitions and prerequisites. Readers who have been already exposed to the subject may skip this part and proceed directly to Section 3.

2.1. Graphoids

Throughout the paper V will denote a finite non-empty set of attributes. These attributes will be represented in the sequel by vertices of graphs and will mainly represent random variables.

Assuming that $X, Y \subseteq V$ the juxtaposition XY will often denote the union $X \cup Y$. A singleton subset $\{v\}$ of the set of attributes V will be denoted by v. We will often deal with triplets (X, Z, Y) of disjoint subsets of V. The set of all such triplets will be denoted by $\mathcal{T}(V)$ and every single such triplet will be called a *triplet over* V.

Let I be a subset of $\mathcal{T}(V)$. We shall sometimes denote the fact that the triplet (X, Z, Y) is in I by $I(X, Z, Y)$. A *graphoid* over V is a set of triplets over V satisfying the following properties. Sometimes we will refer to those properties as axioms.

(0)	$I(\emptyset, Z, Y)$	Trivial property
(1)	$I(X, Z, Y) \Rightarrow I(Y, Z, X)$	Symmetry
(2)	$I(X, Z, YW) \Rightarrow I(X, Z, Y) \wedge I(X, Z, W)$	Decomposition
(3)	$I(X, Z, YW) \Rightarrow I(X, ZY, W)$	Weak Union
(4)	$I(X, ZY, W) \wedge I(X, Z, Y) \Rightarrow I(X, Z, YW)$	Contraction
(5)	$I(X, ZY, W) \wedge I(X, ZW, Y) \Rightarrow I(X, Z, YW)$	Intersection

Given a subset I of $\mathcal{T}(V)$ the graphoid closure of I, to be denoted by $gr(I)$, is understood to be the class of all triplets over V that can be derived from triplets in I by consecutive application of the graphoid properties. Trivially $gr(I)$ is a graphoid.

The relation of conditional relevance with respect to probability theory was defined by Lauritzen [2]. One can interpret conditional irrelevance as conditional independence. Given a joint probability distribution P over a (finite) set of random variables V, the random variables X and Y are irrelevant when Z is known if $P(\mathbf{x}\,\mathbf{y}|\mathbf{z}) = P(\mathbf{x}|\mathbf{z}) \cdot P(\mathbf{y}|\mathbf{z})$ for all possible values $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of random variables X, Y, Z. We will say that a relation $I \subseteq \mathcal{T}(V)$ is *induced by a distribution* P over V if a triplet (X, Z, Y) is in I if and only if X, Z and Y satisfy the above relation.

The well-known fact [4] is that any relation I induced by a probability distribution satisfies the properties (0) - (4) above, and if the distribution P is strictly positive then the induced

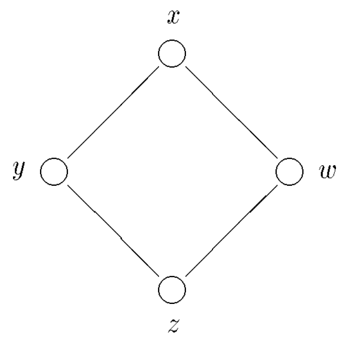


Figure 1. Non-chordal undirected graph.

relation is a graphoid. It has been shown, however, by Studený [7] that the above set of axioms is not complete for relations induced by probabilistic distributions.

Remark 2.1. Triplets of the form $(\emptyset, Z, Y)$ and $(X, Z, \emptyset)$ will be called 'trivial' and could be omitted throughout the paper. But we decided to incorporate them for the sake of completeness.

2.2. Representation by Undirected Graphs

Consider an Undirected Graph (UG) as a way of representation of an irrelevance relation $I \subseteq \mathcal{T}(V)$. Supposing that G is an UG over V (that is a graph having V as its set of vertices), a triplet $(X, Z, Y) \in \mathcal{T}(V)$ is *represented* in G if every path in G from a vertex in X to a vertex in Y is intercepted by a vertex in Z (or equivalently, the set Z is a cutset between X and Y). Of course, if either X or Y is empty, then no such path exists and the triplet (X, Z, Y) is represented in G trivially. The set of triplets represented in G is denoted by $I(G)$. Consider for example the graph shown in Figure 1. The two vertices x and z separate between y and w and therefore the triplet (y, xz, w) is represented in the graph. In addition, the vertices y and w separate between x and z and therefore the triplet (x, yw, z) is represented as well. No other triplet (except for symmetrical images of these two triplets and trivial triplets) is represented in the graph. Thus

$$I(G) = \{\ (y, xz, w), (x, yw, z) + \text{ their symmetrical images } + \text{ trivial triplets }\}.$$

Pearl and Paz [4] gave a characterization of the properties of ternary relations induced by UGs by means of properties of graphoid type. A relation can be represented by an UG if and only if it satisfies the following mutually independent axioms.

(0)	$I(\emptyset, Z, Y)$	Trivial Property
(1)	$I(X, Z, Y) \Rightarrow I(Y, Z, X)$	Symmetry
(2)	$I(X, Z, YW) \Rightarrow I(X, Z, Y) \wedge I(X, Z, W)$	Decomposition
(5)	$I(X, ZY, W) \wedge I(X, ZW, Y) \Rightarrow I(X, Z, YW)$	Intersection
(6)	$I(X, Z, Y) \Rightarrow I(X, ZW, Y),\ W \subseteq V \setminus XYZ$	Strong Union
(7)	$I(X, Z, Y) \Rightarrow I(X, Z, w) \vee I(w, Z, Y),\ w \in V \setminus XYZ$	Transitivity

Remark 2.2. 1. The symbol w in (7) denotes a singleton element of V.

2. The properties above are clearly valid for the set of triplets represented by an UG. Axiom (7) is a contrapositive form of connectedness transitivity, stating that if X is connected to a vertex w and w is connected to Y then X is connected to Y. Axiom (6) states that if Z is a vertex cutset separating X from Y, then adding more vertices W to Z leaves X and Y still separated. Axiom (5) states that if X is separated from Y with W removed and X is separated from W with Y removed, then X must be separated from both Y and W.
3. The Strong Union axiom (6) implies with help of (2) the Weak Union axiom (3) from Section 2.1. Similarly, (5) and (6) imply the Contraction axiom (4) and also the converse of Axiom (2) which is

   (8) $I(X, Z, Y) \wedge I(X, Z, W) \Rightarrow I(X, Z, YW)$ Composition

   meaning that I is completely defined by the set of triplets (a, Z, b) in which a and b are singleton elements of V.

Since the properties (0), (1), (2), (5), (6), (7) together imply the properties (0), (1), (2), (3), (4), (5) the collection of triplets represented in an UG is a graphoid. On the other hand, these two systems of axioms are not equivalent. Consider, for example, the attribute set $V = \{a, b, c\}$ and the graphoid I over V consisting of the triplets $(a, \emptyset, c), (c, \emptyset, a)$ and trivial triplets. It does not satisfy (6). Note that a similar situation occurs in the well-known example with two coins and a bell from the book [5]. Therefore, there are graphoids which cannot be induced by UGs.

A triplet t over a set of attributes V of the form $(a, V \setminus \{a, b\}, b)$ where $a, b \in V$ are distinct, will be called a *simple and saturated* triplet. If G is an UG over V, then such a triplet is represented in G if and only if $\{a, b\}$ is not an edge in G. The claim below can be proved on basis of Theorem 3 from [4].

Claim 2.1. Let G be an undirected graph over V, M be a graphoid over V. If all simple and saturated triplets represented in G are in M, then $I(G) \subseteq M$.

The advantage of graphs for graphoid representation is evident. The representation of a graph requires a polynomial number of bits in the number of its vertices, but the number of triplets which can be represented by it is usually exponential.

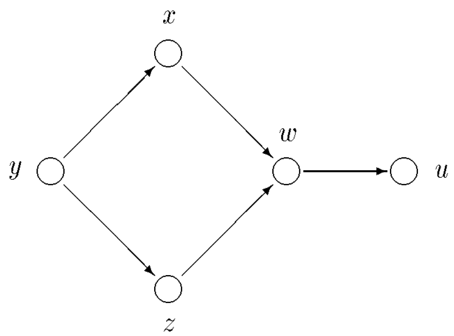


Figure 2. Directed acyclic graph.

2.3. Representation by Directed Acyclic Graphs

A second way of representing graphoids is by Directed Acyclic Graphs (DAGs). The definition of the representation of a triplet in such a graph is more complex and it takes into consideration the possibility of directing the arcs. There are three ways that a pair of arrows may meet at a vertex:

- tail to tail, $x \leftarrow z \rightarrow y$,
- head to tail, $x \rightarrow z \rightarrow y$,
- head to head, $x \rightarrow z \leftarrow y$.

Definition 2.1. 1. Two arrows meeting head to tail, or tail to tail at node u are said to be *blocked* by a set Z of vertices if u is in Z.

2. Two arrows meeting head to head at node u are *blocked* by Z if neither u nor any of its descendants is in Z.
3. An undirected path π in a DAG G is said to be *d-separated* by a subset Z of the vertices if at least one pair of successive arrows along π is blocked by Z.
4. Let X, Y and Z be three disjoint sets of vertices in a DAG G. Z is said to *d-separate* X from Y if all paths between X and Y are d-separated by Z.

For example, in the graph shown in Figure 2, the triplet (z, y, x) is represented as the set $\{y\}$ d-separates between the vertices z and x. On the other hand, the triplets (z, w, x) and (z, yu, x) are not represented in the graph.

This way of graphoid representation has limitations as well. It has been shown by Pearl and Verma [6] that a necessary (but not sufficient) condition for a graphoid to be induced by a DAG is that it satisfies the following independent properties:

(0)	$I(\emptyset, Z, Y)$	Trivial property
(1)	$I(X, Z, Y) \Rightarrow I(Y, Z, X)$	Symmetry
(2a)	$I(X, Z, YW) \Leftrightarrow I(X, Z, Y) \wedge I(X, Z, W)$	Composition - Decomposition
(3)	$I(X, Z, YW) \Rightarrow I(X, ZY, W)$	Weak union
(4)	$I(X, ZY, W) \wedge I(X, Z, Y) \Rightarrow I(X, Z, YW)$	Contraction
(5)	$I(X, ZY, W) \wedge I(X, ZW, Y) \Rightarrow I(X, Z, YW)$	Intersection
(9)	$I(X, Z, Y) \wedge I(X, Zw, Y) \Rightarrow I(X, Z, w) \vee I(w, Z, Y)$	Weak transitivity
(10)	$I(x, zw, y) \wedge I(z, xy, w) \Rightarrow I(x, z, y) \vee I(x, w, y)$	Chordality

Lower case letters stand for singleton elements of V. These properties imply but are not equivalent to the graphoid axioms. Therefore, there are graphoids which cannot be induced by DAGs.

2.4. Comparison of the Two Approaches

As we have seen in the previous two sections, both ways of graphoid representation, by UGs and DAGs, have limitation. There are graphoids that cannot be represented by either one of them. Moreover, no one of the two classes of models is stronger than the other. The example of a graphoid mentioned in Section 2.2 (after Remark 2.2) can be represented by a DAG with three vertices a, b, c and arrows from a to b and from c to b. Thus, the triplet $(a, \emptyset, c)$ is represented, but the triplet (a, b, c) is not represented in that DAG. As mentioned before, this graphoid cannot be represented by an UG.

On the other hand, the graphoid represented by the diamond shaped graph in Figure 1 cannot be represented by a DAG. The graph is non-chordal, and the represented graphoid does not satisfy the Chordality axiom (10).

2.5. The Chain Graph Models

A class of models that generalizes both UG models and DAG model, the class of Chain Graph (CG) models, was used by Frydenberg [1]. We shall provide a brief discussion of this approach model in Section 6.

3. Annotated Graphs

3.1. Definitions and Notation

The following definitions will be required in the sequel.

An *element* over a set of attributes V is a couple $k = (D(k), R(k))$ of disjoint subsets of V, where $D(k)$ is either the empty set or a two-element subset of V and is called the *domain* of the element. The second entry of the element, $R(k)$, is called the *range* of the element. An element is *degraded* if its domain is empty, $D(k) = \emptyset$, otherwise it is *non-degraded*. An element is called *void* if its range is empty, $R(k) = \emptyset$. The collection of all elements over V will be denoted by $E(V)$. Supposing that $K \subseteq E(V)$ the symbol $R(K)$ will denote the union $\bigcup\{R(k); k \in K\}$. An *annotated graph* over V is a couple (G, K) where G is an undirected graph over V and K is a subset of $E(V)$.

Let G be an undirected graph over V. By a *non-trivial* path in G we understand a path connecting at least 3 nodes. For a non-empty subset T of V we define the *restricted graph* of G to T (or in short the restriction of G to T) as a graph over T, denoted by G^T, whose edges are determined by the following requirement: (u, v) is an edge in G^T if there exists a path in G between u and v which is outside $T \setminus \{u, v\}$ (or equivalently through the set of vertices $\{u, v\} \cup (V \setminus T)$).

Remark 3.1. If $\{u, v\} \subseteq T$ is an edge in G, then it forms a (trivial) path between u and v of length 1 which is evidently outside $T \setminus \{u, v\}$, and therefore (u, v) is an edge in G^T. But there may be edges in G^T which are not edges in G. Thus, in general, we can only say that the classic induced subgraph of G for T (usually denoted by G_T) is only a subgraph of the restricted graph G^T (nothing more).

The significance of the restricted graph is explicated by the following lemma.

Lemma 3.1. *Let G be an undirected graph over a set of attributes V and let T be a non-empty subset of V. Then a triplet (X, Z, Y) over T is represented in G if and only if it is represented in the restricted graph G^T.*

Proof:
The set of triplets I over T represented in G satisfies the properties $(0), (1), (2), (5), (6), (7)$. Therefore, I can be perfectly represented by a graph over T (i.e. by a graph representing all the triplets in I and only those triplets - see Section 2.2). It was shown in [4] that such a graph is uniquely determined by its subset of triplets of the form $(a, T \setminus \{a, b\}, b)$. It is easy to see and it is left to the reader to show that the triplets of the above form are represented in the restricted graph G^T if and only if they are represented in G. □

Let $k \in E(V)$ be an element over V and $\emptyset \neq T \subseteq V$. In case $D(k) \subseteq T$ we define the *restricted element* of k to T (or shortly the restriction of k to T), denoted by k^T, as an element over T with $D(k^T) = D(k)$ and $R(k^T) = R(k) \cap T$. Given that (G, K) is an annotated graph over V and T is a non-empty subset of V, the *restricted annotated graph* to T is the graph (G^T, K^T) where $K^T = \{l \in E(T);$ there exists $k \in E(V)$ with $k^T = l\}$.

A *nest* of undirected graphs is a sequence $F_1, \ldots, F_n, n \geq 1$ of undirected graphs such that F_i is a subgraph of F_{i+1} for $i = 1, \ldots, n-1$.

The rest of this paper is devoted to two basic polynomial algorithms and the proof of their correctness. The first algorithm, the membership algorithm, defines the semantics of annotated graphs, i.e. a triplet is represented in a given annotated graph if and only if the membership algorithm when applied on that graph results in a “yes” answer. It will be assumed however that the annotated graphs processed by the algorithm have certain properties and the annotated graphs satisfying those properties will be called regular annotated graphs. It will be shown, in Section 4 that the relations represented in regular annotated graphs via the membership algorithm are graphoid relations. The second algorithm, the annotation algorithm, creates an annotated graph out of a nest of graphs. It will be shown in Sections 5 and 7 that the resulting annotated graph is regular, and that the graphoid relation represented by it is equal to the graphoid closure of the relations represented by the individual graphs in the nest. While the algorithms themselves are quite simple, the proof of their correctness is long and intricate. We choose therefore, for the benefit of the reader, to describe the algorithms first and postpone the proofs to the subsequent sections.

3.2. Membership Algorithm

As mentioned in the previous section, the annotated graphs input to the algorithm will be assumed to be "regular". It is not necessary at this point to define the regularity conditions, and we will do this in the sequel. It will also be shown in the sequel that regular annotated graphs have the following properties (see Lemma 4.1 in Section 4.1). If (G, K) is a regular annotated graph, then $\forall k, l \in K$, $D(k) \cap R(l) \neq \emptyset$ implies that $R(k) \subset R(l)$. Furthermore, the binary relation "$\preceq$" over K defined by $l \preceq k$ if $[D(k) \cap R(l) \neq \emptyset$ or $k = l]$ is a partial ordering on K. Before presenting the algorithm itself, we need also the following definitions.

Definition 3.1. Suppose that (G, K) is a regular annotated graph, and $k, l \in K$. We say that k *dominates* l and write $l \prec k$ or $k \succ l$ if $D(k) \cap R(l) \neq \emptyset$. Observe that $l \prec k$ implies $l \neq k$ since $R(k) \cap D(k) = \emptyset$ for every element k over V. An element is called a *dominant element* of K if there is no element in K which dominates k (equivalently, k is a maximal element of K with respect to the partial ordering $\preceq$ mentioned above).

Remark 3.2. Note that, it follows from the definition, a degraded element cannot dominate another element, but it can be dominated by other elements. On the other hand, degraded elements may be dominant (this happens when they are not dominated).

Definition 3.2. Let (G, K) be a regular annotated graph with $K \neq \emptyset$. We say that a sequence $\omega = (k_1 \ldots, k_n), n \geq 1$ of all elements in K is a *scenario* for K if the following three conditions hold.

(a) Whenever $k_i \succ k_j$ then $i < j$.
(b) Every element of K is included in ω.
(c) Non-degraded elements are not repeated in ω.

It follows from the above conditions that the first element of a scenario for K must be a dominant element of K. Note that degraded elements can be repeated in a scenario.

The membership algorithm

0. *Input:* (G, K) a regular annotated graph over a finite non-empty set of vertices V, and $t = (X, Z, Y)$ a triplet over V.
1. *Initiation:* Construct a scenario ω for K such that every element of K is included in it exactly once, and such that any element $l \in K$ with $R(l) \cap XYZ = \emptyset$ precedes in ω all elements $k \in K$ with $R(k) \cap XYZ \neq \emptyset$. { This is possible due to the fact that $\preceq$ is a partial ordering and $R(l) \cap XYZ = \emptyset \neq R(k) \cap XYZ$ implies that $\neg\{ k \succ l \}$ as otherwise $R(k) \subset R(l)$. }
2. *Deletion:* Remove from K all elements $r \in K$ with $R(r) \cap XYZ \neq \emptyset$ and at the same time cancel those elements in ω. { Thus, ω is shortened by cutting off all elements in the sequence after the last element $l \in K$ with $R(l) \cap XYZ = \emptyset$. }
3. *Testing:* If K is empty, then test whether t is represented in the resulting undirected graph and halt with "yes" or "no", depending on the result of the test.
4. *Processing:* K is not empty. Pick the first element $p \in K$ in ω { p is a dominant element } and perform the following 3 steps. Let us put $S = R(p)$.

   4.1 *Degradation:* For every non-degraded element $s \in K$ such that *there exists a non-trivial path in G between the nodes of $D(s)$ through $D(s) \cup S \setminus R(s)$*, replace the element $s = (D(s), R(s))$ by its degraded version $(\emptyset, R(s))$ both in K and ω. If $\tilde{s} = (\emptyset, R(s))$ was already in K before Step 4.1, just remove s from K and degrade it in ω. { This step may result in repetition of degraded elements in ω even though such repetition is not possible in the set K. }

   4.2 *Restriction:* Replace the annotated graph by its restriction to $V \setminus S$ and at the same time replace in ω every element by its restriction to $V \setminus S$. { Note that in this step the processed element p is changed into an void element. }

   4.3 *Reduction:* For every non-degraded void element l in K (i.e. whenever $R(l) = \emptyset \neq D(l)$) remove the edge connecting the vertices in $D(l)$ from the graph G, if such an edge exists. Then remove all void elements (including degraded ones) from K and cancel them in ω. { The processed element $p \in K$ is deleted in this step. After this step all the elements in K have non-empty range. }
5. Go to 3.

3.3. Annotation Algorithm

Let G be an undirected graph over V and let $k = (\{a, b\}, U)$ be a non-degraded element over V. The symbol $\tau(a, b|U||G)$ will denote the set of all vertices $y \in V \setminus \{a, b\}$ such that for both $x \in \{a, b\}$, there is a path in G between x and y through $\{x\} \cup U$. The basis of the annotation algorithm is the following annotation procedure.

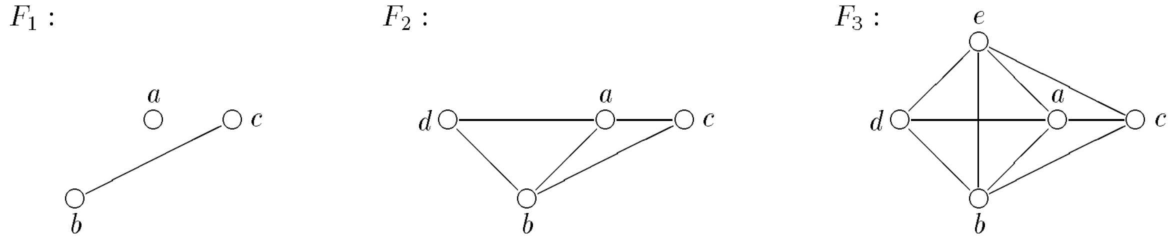


Figure 3. A nest of UGs.

Definition 3.3. Let (H, L) be an annotated graph over V with no degraded element. Let F be an undirected graph over $V \cup B$, where $V \cap B = \emptyset$, such that H is a subgraph of F. The annotated graph (G, K) over $V \cup B$ denoted by Annot $((H, L) : F)$ is defined as follows. { Recall that degraded elements are elements of the form $(\emptyset, U)$, where $U \subseteq V$.}

A1. G is derived from F by removing from F all edges (u, v) such that $u, v \in V$, (u, v) is not an edge in H and $\tau(u, v|B||F) = \emptyset$. Set $K = \emptyset$.

A2. Insert 'new' elements in K as follows: for every pair of vertices $u, v \in V$, $u \neq v$, such that (u, v) is not an edge in H and $U = \tau(u, v|B||F) \neq \emptyset$, create a (non-degraded) element $(\{u, v\}, U)$ and insert it into K.

A3. Add to K elements created from elements in L by 'expanding' as follows. For every element $(\{u, v\}, W) \in L$ add to K the expanded element $(\{u, v\}, T)$ where $T = \tau(u, v|W \cup B||F)$. { Note that in standard case of a regular annotated graph (H, L) it holds $W \subseteq T$ and therefore we are entitled to say that $(\{u, v\}, W)$ is expanded into $(\{u, v\}, T)$. }

The annotation algorithm

Input: A nest of undirected graphs $F_1, F_2 \ldots, F_n$, $n \geq 1$.
Start from (G_1, K_1) with $G_1 = F_1$ and $K_1 = \emptyset$.
Construct $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$ for $i = 2, \ldots, n$.
Output: $(G, K) = (G_n, K_n)$.

As mentioned before, it will be shown in the sequel that the annotated graphs, obtained by the annotation algorithm, are regular, that the relations represented (via the membership algorithm) by regular annotated graphs are graphoid relations and that the graphoid closure of the relations represented by the individual UG's in a nest of UG' s is identical to the graphoid relation represented by the regular annotated graph derived from the given nest by the annotation algorithm. We will also show that not every regular annotated graph representing a graphoid relation can be derived from a nest of UG's.

3.4. Example

Consider the nest of UG's given in Figure 3. Every F_i in the nest is a subgraph of F_{i+1}, $i = 1, 2$ as requested. Applying the annotation algorithm to the above graphs, we get (K_i is the annotation of G_i):

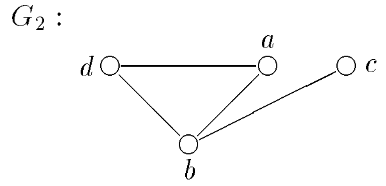


Figure 4. The second iteration of the annotation algorithm.

- First iteration (G_1, K_1) with $G_1 = F_1$ and $K_1 = \emptyset$.
- Second iteration (G_2, K_2) with G_2 as in Figure 4 and $K_2 = \{(\{a,b\},\ \{d\})\}$. The edge (a,c) is removed from F_2 to get G_2 since $\tau(a,c|\{d\}\|F_2) = \emptyset$ (Step A1), the element $(\{a,b\},\ \{d\})$ is added to K_2 since $\tau(a,b|\{d\}|F_2) = \{d\}$ (Step A2).
- Third iteration (G_3, K_3) with $G_3 = F_3$ and

$$K_3 = \{\,(\{a,b\},\{d,e\}),\ (\{a,c\},\{e\}),\ (\{c,d\},\{e\})\,\}\,.$$

  No pair (u,v) such that (u,v) is an edge in F_3 but not in G_2 satisfies $\tau(u,v|\{e\}\|F_3) = \emptyset$, and therefore no edge is removed from F_3 in order to get G_3. Since $\tau(a,c|\{e\}\|F_3) = \{e\}$ and $\tau(c,d|\{e\}\|F_3) = \{e\}$ the elements $(\{a,c\},\ \{e\})$ and $(\{c,d\},\ \{e\})$ are added to K_3 (Step A2). Moreover, $\tau(a,b|\{d,e\}\|F_3) = \{d,e\}$ and therefore the element $(\{a,b\},\ \{d\})$ of K_2 is expanded into the element $(\{a,b\},\ \{d,e\})$ in K_3 (Step A3).

Notice that, in K_3, the element $(\{c,d\},\ \{e\})$ dominates the element $(\{a,b\},\ \{d,e\})$. Thus, the only dominant elements of K_3 are $(\{a,c\},\ \{e\})$ and $(\{c,d\},\ \{e\})$. Suppose we want to test whether the triplet (a,bd,c) is represented in (G_3, K_3) then we go through the following steps of the membership algorithm:

- $\omega = (\{a,c\},\ \{e\}),\ \ (\{c,d\},\ \{e\}),\ \ (\{a,b\},\ \{d,e\})$ (Step 1 - Initiation).
- Remove the element $(\{a,b\},\ \{d,e\})$ from K_3 and from ω (Step 2 - Deletion).
- The condition of Step 3 does not hold, as $K_3 \neq \emptyset$.
- Process the element $(\{a,c\},\ \{e\})$ (Step 4).

  4.1 Does not apply.

  4.2 The graph G_3 is changed into the complete graph G_3' over $\{a,b,c,d\}$ and the ranges of both remaining elements are set to $\emptyset$. Thus, the next iteration is (G_3', K_3') with $K_3' = \{\,(\{a,c\},\emptyset),(\{c,d\},\emptyset)\,\}$.

  4.3 The edges (a,c) and (c,d) are now removed from G_3' which is then transformed into $G_3'' = G_2$. All elements are removed from K_3' to get $K_3'' = \emptyset$ and cancelled in the scenario.

- Returning to Step 3 we find that the triplet (a,bd,c) is represented in $G_3'' = G_2$ so that the algorithm halts with a “yes” answer.

Notice that the given triplet is not represented in any of the individual UG's in the original nest, but as a result of the "yes" answer given by the membership algorithm, we know that it is represented in the graphoid closure of the relations represented by F_1, F_2 and F_3.

Remark 3.3. The reader can verify that the graphoid represented by the annotated graph (G_3, K_3) cannot be represented by an UG or a DAG since axioms (6) and (9) are not fulfilled. Another example of such an annotated graph will be given in Section 5.3.

4. Proofs

As mentioned earlier, this section of the paper is devoted to proofs. The order of the proofs does not follow the order of the exposition, and the most complex parts are postponed to Section 7. We provide first the definition of regularity for annotated graph, then we show that regular annotated graphs represent graphoids and can be tested, polynomially, for membership of individual triplets.

4.1. Regular Annotated Graphs

The concept of annotated graphs introduced in the previous section is too general. In fact, we will restrict our attention to a special class of annotated graphs, which satisfy certain regularity conditions. The conditions seem technical at first sight, but they express important general properties shared by the annotated graphs we deal with.

Definition 4.1. Suppose that (G, K) is an annotated graph over V. We say that an edge (u, v) in (G, K) is K-*durable* (or simply *durable*) if there is no element in K whose domain is $\{u, v\}$. We say that an annotated graph (G, K) is *regular* if it satisfies the following three conditions.

(R1) $\forall k \in K,\ \forall u \in R(k),\ \forall v \in D(k)$,
there exists a path in G between u and v through $\{v\} \cup R(k)$ composed of K-durable edges which is completely outside $R(K_u)$ where $K_u = \{l \in K;\ u \notin R(l)\}$.

(R2) $\forall k, l \in K$ such that $D(k) = D(l) \neq \emptyset$ there exists $q \in K$ with $D(q) = D(k)$ and $R(q) = R(k) \cup R(l)$.

(R3) $\forall k, l \in K,\ \forall$ path $w_1, \ldots, w_n,\ n \geq 2$ in G through $\{w_1, w_n\} \cup R(k)$ such that $w_1 \in R(l) \setminus R(k)$, $w_n \in V \setminus R(k)$, and (w_1, w_n) is not a K-durable edge in G, there are indices $1 \leq i < j \leq n$ and $q \in K$ such that $D(q) = \{w_i, w_j\}$ and $\{w_h : i < h < j\} \subseteq R(q)$.

Remark 4.1. The condition requiring that (w_1, w_n) is not a K-durable edge means that either (w_1, w_n) is not an edge or, if it is, there is an element $r \in K$ such that $D(r) = \{w_1, w_n\}$.

The conditions (R1), (R2), (R3) are independent of each other. For example, to show that (R1) (R2) $\not\Rightarrow$ (R3) consider the annotated graph in Figure 5, to show that (R1) (R3) $\not\Rightarrow$ (R2) take the annotated graph in Figure 6, and to show that (R2) (R3) $\not\Rightarrow$ (R1) use the annotated graph in Figure 7.

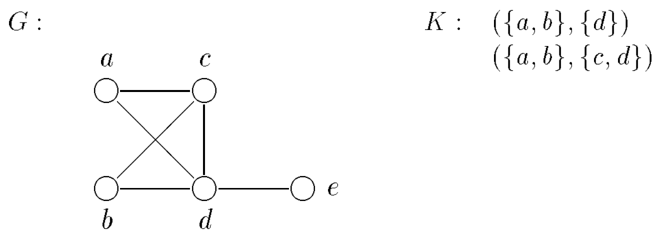


Figure 5. Annotated graph without (R3) property.

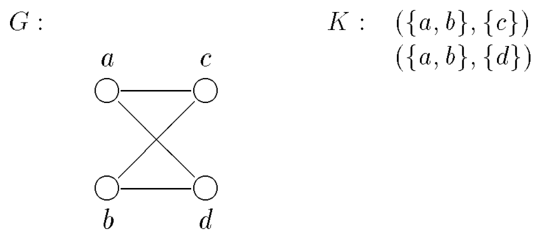


Figure 6. Annotated graph without (R2) property.

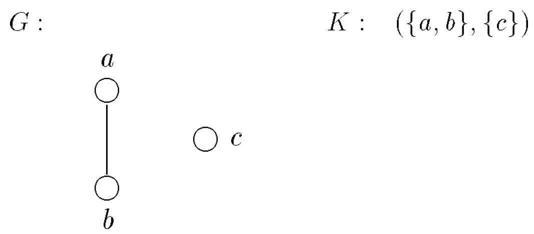


Figure 7. Annotated graph without (R1) property.

Lemma 4.1. *Let* (G, K) *be an annotated graph satisfying* (R1). *Then*

(R4) $\forall k, l \in K \quad D(k) \cap R(l) \neq \emptyset$ *implies that* $R(k) \subset R(l)$.

Furthermore, the binary relation $\preceq$ *over* K *defined by* $l \preceq k$ *if* $[D(k) \cap R(l) \neq \emptyset$ *or* $k = l]$ *is a partial ordering on* K. *Notice that* $l \preceq k$ *implies* $R(k) \subseteq R(l)$.

Proof:
Assume that $k, l \in K, D(k) \cap R(l) \neq \emptyset$. Since $D(k) \cap R(k) = \emptyset$ one has $\emptyset \neq D(k) \cap R(l) \subseteq R(l) \setminus R(k)$ and to show $R(k) \subset R(l)$ it remains to verify $R(k) \subseteq R(l)$. Suppose for a contradiction that $R(k) \setminus R(l) \neq \emptyset$. Then choose $u \in R(k) \setminus R(l)$, $v \in D(k) \cap R(l)$ and apply (R1) to find a path between u and v outside $R(K_u)$. Since $u \notin R(l)$ it should be outside $R(l)$, but v, its last node, is in $R(l)$ by our assumption which is a contradiction. Thus, (R4) was verified and $l \preceq k$ implies $R(k) \subseteq R(l)$. Evidently, the relation $\preceq$ is reflexive. To verify that it is antisymmetric, consider $k, l \in K$ such that $l \preceq k$ and $k \neq l$. Then $D(k) \cap R(l) \neq \emptyset$ implies by $(R4)$ that $R(k) \subset R(l)$. Thus, $R(l) \setminus R(k) \neq \emptyset$ forces that $\neg(k \preceq l)$. To prove transitivity consider $k, l, s \in K$ with $s \preceq l \preceq k$. We must show that $s \preceq k$. But $s \preceq l$ implies $R(l) \subseteq R(s)$ which together with $D(k) \cap R(l) \neq \emptyset$ means that $D(k) \cap R(s) \neq \emptyset$, what is needed. □

The following lemma deals with the concept of dominance introduced in Definition 3.1.

Lemma 4.2. *Let* (G, K) *be an annotated graph satisfying* (R2) *and* (R3). *Suppose that* $p, s \in K$ *are given such that* p *is a dominant element of* K *and* s *dominates some element of* K *(no matter which one). Then there exists* $q \in K$ *with* $D(q) = D(s), R(s) \subseteq R(q)$ *and there is no non-trivial path in* G *between the nodes of* $D(q)$ *through* $D(q) \cup R(p) \setminus R(q)$.

Proof:
Clearly s is a non-degraded element. Put $K_* = \{r \in K;\ D(r) = D(s)\}$. Since K_* is finite and non-empty ($s \in K_*$) by consecutive application of (R2) find $q \in K_*$ with $R(q) = R(K_*)$. Therefore $R(s) \subseteq R(q)$. Suppose by contradiction that there exists a path $w_1, ..., w_n$, $n \geq 3$ in G between the nodes of $D(q) = \{w_1, w_n\}$ through $D(q) \cup (R(p) \setminus R(q))$. Since $s \in K$ dominates some $l \in K$, we can assume, without loss of generality, that $w_1 \in R(l) \setminus R(p)$ and $w_n \in V \setminus R(p)$ (note that $D(q) \cap R(p) = \emptyset$ since p is a dominant element of K). Then one can use (R3) to find $k \in K$ with $D(k) = \{w_i, w_j\}$ and $\{w_h; i < h < j\} \subseteq R(k)$ for some $1 \leq i < j \leq n$. Since p is a dominant element, we have that $i = 1$ and $j = n$ necessarily. Thus $k \in K_*$ and $\emptyset \neq \{w_2, ..., w_{n-1}\} \subseteq R(k) \subseteq R(K_*) = R(q)$. This contradicts the assumption that $\{w_2, ..., w_{n-1}\} \subseteq R(p) \setminus R(q)$. □

4.2. Properties of the Membership Algorithm

We refer now to the algorithm presented in Section 3.2. We will show that the algorithm preserves the regularity of the annotated graphs from iteration to iteration. In the initiation step and before the first iteration the elements in the set K at input are ordered into a special sequence called *scenario*. After this sequence is set, it becomes a free parameter of the algorithm and the algorithm becomes fully deterministic. No further free choices are allowed in subsequent iterations, and the sequence of elements constructed at initiation is no longer dependent on the tested triplet t.

We would like to mention here that the formulation of the algorithm as given in Section 3.2 renders the algorithm more complex than necessary. This formulation is needed in order to simplify some of the proofs. Some modifications, rendering the algorithm more efficient, will be shown in the sequel. Let us add a few remarks concerning the given definition of the algorithm.

1. Notice that the processing of an element $p \in K$ in Step 4 may also eliminate other elements of K.
2. Notice that all parts of Step 4 depend on $S = R(p)$ only! It is immaterial whether p is degraded or not. In fact, for every regular annotated graph (G, K) over V and every set $S \subseteq V$ with $D(k) \cap S = \emptyset$ for every $k \in K$, the concept of *processing of the set S* can be introduced. This means that the Steps 4.1, 4.2 and 4.3 are performed while the operations with ω are ignored. This formal point of view will be useful later.
3. The reader may consider it superfluous to construct a scenario consisting of all elements of K, in Step 1, since the end part of that sequence is immediately cut off in Step 2. The suitability of this formal step will also become clear in the sequel (proof of Theorem 4.2).

Observation 4.1. After every step of the membership algorithm, (G, K) is an annotated graph and ω satisfies the first two conditions in the definition of a scenario.

Proof:
The only doubts can occur in the restriction step 4.2. But, since p was a dominant element of K before this step (in fact, p was a dominant triplet of K already before the degradation step 4.1 and degradation preserves that situation), we know that the set $V \setminus R(p)$ includes all domains of elements in K before restriction. Therefore, for arbitrary $l, s \in K$ we know that $l^{V\setminus R(p)}$ dominates $s^{V\setminus R(p)}$ if and only if l dominates s, and the obtained sequence ω of restricted elements will satisfy the conditions (a) and (b) in the definition of the scenario. □

Observation 4.2. If a triplet $\tilde{t}$ over the set of vertices of the last iteration is already represented in an iteration (G, K, ω) of the membership algorithm (that means $\tilde{t}$ is represented in the undirected graph G generated at that iteration), then it is represented in its last iteration, too.

Proof:
The undirected graph is changed only in Steps 4.2 and 4.3 of the algorithm. By Lemma 3.1 the restriction step 4.2 preserves representation of a triplet $\tilde{t} = (\tilde{X}, \tilde{Z}, \tilde{Y})$ with $\tilde{X}\tilde{Y}\tilde{Z} \subseteq V \setminus R(p)$. In the reduction step 4.3 only some edges can be possibly removed from the undirected graph. This change also preserves representation of a triplet in the graph. □

Remark 4.2. It follows from the previous observation that we can make the membership algorithm more efficient by modifying Step 3 as follows:

3. *Testing (modified):* Test whether the triplet t is represented in G. If the answer is "yes", then stop. Otherwise continue untill K is empty.

4.3. Basic Results

Lemma 4.3. *After every step of the membership algorithm, (G, K) is a regular annotated graph and ω is a scenario for K.*

Proof:
Evidently, the described situation holds after Step 1. We need to show that the situation is preserved during Steps 2 and 4. Owing to Observation 4.1, we have only to show that (R1), (R2), (R3) are preserved and that non-degraded elements are not repeated in the sequence ω, after those steps.

Now the deletion step 2 represents only the removal of an element $r \in K$ with $R(r) \cap XYZ \neq \emptyset$, the undirected graph G is not changed. To verify (R1) assume that after the deletion (= Step 2), one has $k \in K$, $u \in R(k)$ and $v \in D(k)$, for some k, u v. Then this was also the case before the deletion. Based on condition (R1) we know that the vertices u and v were connected by a path through $\{v\} \cup R(k)$ made of K-durable edges outside of $R(K_u)$ (before the deletion step). But durable edges, before the deletion remain durable after it, while $R(K_u)$ can only be decreased. Therefore, the condition described in (R1) remains valid after the deletion step 2.

To verify (R2), we notice that if $k, l \in K$ with $D(k) = D(l) \neq \emptyset$ after the deletion step, then the same holds before deletion. Therefore an element $q \in K$ such that $D(q) = D(k)$ and $R(q) = R(k) \cup R(l)$ was present in K before deletion (by (R2)). Now, both $R(k)$ and $R(l)$ do not intersect XYZ and therefore $R(q) \cap XYZ = \emptyset$ implying that q remains in K after Step 2.

To verify (R3) suppose that the premises of (R3) are satisfied after the deletion step 2 (see the formulation of (R3)). One can verify that the premises of (R3) are valid also before deletion, implying the existence of $q \in K$ such that $D(q) = \{w_i, w_j\}$ and $\{w_h, i < h < j\} \subseteq R(q)$ for some $1 \leq i < j \leq n$ (where $w_1, ..., w_n$ is the assumed path). If $k, l \in K$ were the assumed elements before deletion, then one has either $D(q) \cap R(l) \neq \emptyset$ (in case $i = 1$) or $D(q) \cap R(k) \neq \emptyset$ (in case $2 \leq i$). Hence, by Lemma 4.1 (property (R4)) one can derive that $R(q) \subseteq R(k) \cup R(l)$ was valid before the deletion step. Since both $R(k)$ and $R(l)$ do not intersect XYZ, q is preserved in K during Step 2.

Evidently, Step 2 does not create repetitions of elements.

Suppose that in Step 4 an element $p \in K$, which is a dominant element of K, is processed. Then Step 4.1 degrades only some non-degraded elements $s \in K$, namely those for which there exists a non-trivial path in G between nodes of $D(s)$ through $D(s) \cup (R(p) \setminus R(s))$. The undirected graph G is unchanged.

The verification of (R1) after the degradation Step 4.1 can be made in a similar way as it was made in the case of the deletion Step 2: durable edges before degradation remain durable after degradation and $R(K_u)$ is unchanged for every node u.

To verify (R2) suppose that one has $k, l \in K$ with $D(k) = D(l) \neq \emptyset$ after degradation. Then the same situation occurs before degradation, and one can find $q \in K$ with $D(q) = D(k)$ and $R(q) = R(k) \cup R(l)$ (before degradation). Suppose for a contradiction that q is degraded during Step 4.1, which means that there was a non-trivial path in G between the nodes of

$D(q) = D(k)$ through $D(q) \cup (R(p) \setminus R(q))$. Since $R(k) \subseteq R(q)$, the path also went through $D(k) \cup (R(p) \setminus R(k))$, what contradicts the assumption that k is a non-degraded element after the degradation step 4.1. Thus, q can not be changed during degradation.

To verify (R3), suppose that its premises are satisfied after the degradation step 4.1 with respect to (possibly degraded) elements $\tilde{k}, \tilde{l} \in K$. Then the premises of (R3) were also satisfied before degradation and with respect to the original versions $k, l \in K$ (one has either $\tilde{k} = k$ or $\tilde{k}$ is a degraded version of a non-degraded element k, similarly for l and $\tilde{l}$). Thus, (R3) guarantees that a non-degraded element $s \in K$ with $D(s) = \{w_i, w_j\}$ and $\{w_h;\ i < h < j\} \subseteq R(s)$ for some $1 \leq i < j \leq n$ can be found ($w_1, \ldots, w_n$ is the assumed path - see the formulation of (R3)). Now, one can see that s dominates either k or l, before degradation (one can repeat the arguments we used when we verified that (R3) is preserved during Step 2). Thus, the assumptions of Lemma 4.2 for $p, s \in K$ were fulfilled before degradation implying the existence of $q \in K$ with $D(q) = D(s), R(s) \subseteq R(q)$ such that q is not degraded during Step 4.1. Thus, $q \in K$ satisfies both $D(q) = \{w_i, w_j\}$ and $\{w_h; i < h < j\} \subseteq R(q)$ and is preserved during the degradation step.

Evidently, Step 4.1 does not create repetition of non-degraded elements in ω.

The restriction step 4.2 (with respect to a dominant element $p \in K$) represents the restriction of the undirected graph G to $V \setminus R(p)$ and replacement of every element $k \in K$ by its restriction $k^{V \setminus R(p)}$.

To verify the validity of (R1) suppose that we have, after Step 4.2, a restricted element $\tilde{k} \in K^{V \setminus R(p)}$, $u \in R(\tilde{k})$ and $v \in D(\tilde{k})$. Then before restriction there was $k \in K$ such that $k^{V \setminus R(p)} = \tilde{k}$ with $u \in R(k) \setminus R(p)$, $v \in D(k) \subseteq V \setminus R(p)$. By the property (R1) we were able to find a path between u and v in G through $\{v\} \cup R(k)$ composed of K-durable edges which is outside $R(K_u)$. Since $u \notin R(p)$, one has that $R(p) \subseteq R(K_u)$ and all the paths belong to $V \setminus R(p)$. Therefore, the path is also a path in the restricted graph $G^{V \setminus R(p)}$ through $\{v\} \cup R(\tilde{k})$. Moreover, as the domains of elements were not changed by the restriction step, durable edges in $V \setminus R(p)$ remain durable. Also $R(K_u)$ was only diminished during this step. Therefore the property (R1) remains valid also after Step 4.2.

To verify (R2), suppose that after restriction we have two restricted elements $\tilde{k}, \tilde{l} \in K^{V \setminus R(p)}$ with $D(\tilde{k}) = D(\tilde{l}) \neq \emptyset$. Then there exist $k, l \in K$ with $\tilde{k} = k^{V \setminus R(p)}$ and $\tilde{l} = l^{V \setminus R(p)}$. Since $D(k) = D(l) \neq \emptyset$, we can find by (R2) an element $q \in K$ with $D(q) = D(k)$ and $R(q) = R(k) \cup R(l)$. But the restriction of q to $V \setminus R(p)$ results in an element $\tilde{q}$ with $R(\tilde{q}) = R(\tilde{k}) \cup R(\tilde{l})$.

To verify (R3), suppose that the premises of (R3) are satisfied after Step 4.2, that is we have two restricted elements $\tilde{k}, \tilde{l}$ and a path $w_1, ..., w_n, n \geq 2$ in the restricted graph $G^{V \setminus R(p)}$ from $w_1 \in R(\tilde{l}) \setminus R(\tilde{k})$ to $w_n \in V \setminus (R(p) \cup R(\tilde{k}))$ through $\{w_1, w_n\} \cup R(\tilde{k})$ such that either $\{w_1, w_n\}$ is not an edge in $G^{V \setminus R(p)}$, or there exists a restricted element $\tilde{r}$ with $D(\tilde{r}) = \{w_1, w_n\}$. Let $k, l \in K$ be two elements of K (before restriction), such that $\tilde{k} = k^{V \setminus R(p)}$ and $\tilde{l} = l^{V \setminus R(p)}$. One must consider two subcases.

- If the whole path $w_1 \ldots, w_n$ consists of edges in the original graph G, then one can verify that the premises of (R3) for k, l and $w_1, ..., w_n$ are satisfied before Step 4.2 and find by the application of (R3) the required $q \in K$ (see the formulation of (R3)). Then the corresponding restricted element $q^{V\setminus R(p)}$ is the desired element verifying (R3) after Step 4.2.

- On the other hand, if (w_i, w_{i+1}) is not an edge in G for some $1 \le i \le n-1$, then by the definition of the restricted graph, there exists a path $u_1, ..., u_h, h \ge 3$ in G between $u_1 = w_i$ and $u_h = w_{i+1}$ through $\{u_1, u_h\} \cup R(p)$. Moreover, $u_1 = w_i$ either belongs to $R(l) \setminus R(p)$ (if $i = 1$) or to $R(k) \setminus R(p)$ (if $2 \le i \le n-1$). Thus, one can apply (R3) either to the couple $p, l \in K$ and $u_1, ..., u_h$ or to the couple $p, k \in K$ and $u_1, ..., u_h$ to find $q \in K$ with $D(q) = \{u_{i^*}, u_{j^*}\}$ for some $1 \le i^* < j^* \le h$. Since p is a dominant element of K, necessarily $i^* = 1$ and $j^* = h$. Therefore, $D(q) = \{u_1, u_h\} = \{w_i, w_{i+1}\}$. Then the corresponding restricted element $q^{V\setminus R(p)}$ is the desired element verifying (R3) for $w_1, ..., w_n$ with $j = i+1$.

This completes the verification of (R3).

To show that non-degraded elements are not duplicated in ω after the restriction step 4.2, suppose for a contradiction that there are different non-degraded elements $k, l \in K$ (before Step 4.2) whose restrictions to $V \setminus R(p)$ coincide, i.e., $k^{V\setminus R(p)} = l^{V\setminus R(p)}$. Thus, consider the situation before the restriction step and put

$$K_* = \{\, s \in K;\ D(s) = D(k) \ \text{ and } \ R(s) \setminus R(p) = R(k) \setminus R(p) \,\}$$

Evidently, $k, l \in K_*$ and, by consecutive application of (R2), we can find $q \in K$ with $D(q) = D(k)$ and $R(q) = R(K_*)$. Clearly, $q \in K_*$. Since k and l differ but can be interchanged, one can assume, without loss of generality, that $R(l) \setminus R(k) \ne \emptyset$ and one can find and fix some $u \in R(q) \setminus R(k)$.

Now, for both $v_1, v_2 \in D(q) = D(k)$ one can find by (R1) (for q) a path in G between u and v_i $(i = 1, 2)$ through $\{v_i\} \cup R(q)$ which is outside $R(k)$ (since $k \in K_u$!). However, such a path must pass through $\{v_i\} \cup R(p)$ (since $q \in K_*$ implies that $R(q) \setminus R(p) \subseteq R(k)$). So, for both $v_i \in D(q)$ $(i = 1, 2)$ there exists a path in G between u and v_i through $\{v_i\} \cup (R(p) \setminus R(k))$. These two paths can be joined and possibly shortened to obtain a non-trivial path in G between nodes of $D(k) = D(q)$ through $D(k) \cup (R(p) \setminus R(k))$. This situation occurs before Step 4.2, that is after Step 4.1. One can easily see that such a path in G exists also before Step 4.1 which implies that k was necessarily degraded in the degradation Step 4.1 which contradicts the assumption.

The reduction Step 4.3 represents the removal of void elements from K and in G the removal of edges which are possibly domains of those removed elements.

To verify validity of (R1), suppose that $k \in K$, $u \in R(k)$, $v \in D(k)$ after reduction. Then this is the case also before the reduction, and by (R1) one can find a corresponding path in G made of K-durable edges which is outside $R(K_u)$ (before reduction). As Step 4.3 removes only non-durable edges, the path will remain in the graph after the reduction. Since K was reduced, durable edges remain durable and $R(K_u)$ is unchanged during Step 4.3. Therefore, the property (R1) holds also after the reduction step.

To verify (R2) suppose, that after reduction, there are two elements $k, l \in K$ with $D(k) = D(l) \ne \emptyset$. Then this situation was present also before the reduction and, by property (R2), one can find $q \in K$ with $D(q) = D(k)$ and $R(q) = R(k) \cup R(l)$. Now $R(k) \neq \emptyset$ implies that $R(q) \neq \emptyset$ and q is saved during the reduction step.

To verify the validity of (R3), assume that the premises of (R3) are satisfied after the reduction step 4.3 (see the formulation of (R3)). To check that they are also satisfied before the reduction, notice that the only disputable case is that where (w_1, w_n) is not an edge after Step 4.3, although it is an edge before the reduction. Necessarily, $\{w_1, w_n\}$ is a domain of a non-degraded void element before reduction. Thus, by property (R3), applied before reduction, we can find $q \in K$ with $D(q) = \{w_i, w_j\}$ and $\{w_h; i < h < j\} \subseteq R(q)$ for some $1 \leq i < j \leq n$. The case $R(q) = \emptyset$ requires that $j = i + 1$ and the edge (w_i, w_{i+1}) has to be removed during reduction contrary to our assumption that the path $w_1, ..., w_n$ exists in the graph after it. So, necessarily q is saved during Step 4.3.

Evidently, Step 4.3 does not create duplication of elements in ω.

□

In fact, we have shown in the previous proof that the regularity conditions (R1), (R2), (R3) are saved during processing of the set $S = R(p)$ for a dominant element $p \in K$. Since processing of the empty set makes no change (except Step 4.3) one can conclude the following.

Consequence 4.1. Let (G, K) be a regular annotated graph over V. If $S = \emptyset$ or $S = R(p)$ for a dominant element $p \in K$, then the processing of S results in a regular annotated graph over $V \setminus S$.

Lemma 4.4. *Let (G, K) be a regular annotated graph, $p, q, r \in K$ where p, q are dominant elements of K, $R(r) \subseteq R(p) \cup R(q)$, $u, v \in V \setminus R(r), u \neq v$ such that $\{u, v\} \cap (R(p) \cup R(q)) \neq \emptyset$. If (u, v) is an edge in $G^{V \setminus R(r)}$, then (u, v) is an edge in G.*

Proof:
Without loss of generality suppose that $u \in R(q)$ (otherwise one can interchange u and v and also interchange p and q). By the definition of $G^{V \setminus R(r)}$ there exists a path $w_1 \ldots, w_n$, $n \geq 2$ in G with $u = w_1$, $v = w_n$ through $\{w_1, w_n\} \cup R(r)$. Suppose, by contradiction, that $(u, v) = (w_1, w_n)$ is not an edge in G and, by application of (R3) (for $l = q, k = r$), find $s \in K$ with $D(s) = \{w_i, w_j\}$ for some $1 \leq i < j \leq n$. Since $i \leq n - 1$, one has $w_i \in R(q) \cup R(r) \subseteq R(p) \cup R(q)$. This implies that s dominates either p or q, which contradicts the assumption that p and q are dominant. Thus, necessarily, (u, v) is an edge in G. □

Lemma 4.5. *Let (G, K) be a regular annotated graph and ω_1, ω_2 two scenarios for K which differ only in the order of the first two elements $p, q \in K$, that is:*

$$\omega_1 : p, q, s_1, ..., s_n \quad and \quad \omega_2 : q, p, s_1, ..., s_n \;, \quad n \geq 0.$$

(it is understood that if $n = 0$ then no element of K succeeds p and q both in ω_1 and in ω_2). Then the application of two iterations of the membership algorithm to (G, K, ω_1) and to (G, K, ω_2) results in the same intermediary output (G_*, K_*, ω_*).

Proof:

We need to show that after the processing of p and q the resulting annotated graph (G_*, K_*) is the same for both ω_1 and ω_2. The fact that the resulting scenario ω_* coincides is evident, as the mutual order of the remaining elements is not changed, with some of the elements possibly degraded, restricted or removed.

If the processing of p and q are performed on ω_1, then we must consider the following consecutive six steps: degradation for p, restriction for p, reduction (for p), degradation for $q^{V\setminus R(p)}$, restriction for $q^{V\setminus R(p)}$, reduction (for $q^{V\setminus R(p)}$).

Note that if $R(q) \subseteq R(p)$ then the last 3 steps do not apply since the element $q^{V\setminus R(p)}$ is already cancelled in the reduction step (for p). Similarly, if the iterations are performed on ω_2 we must consider the six steps derived from the above six steps when p is interchanged with q. Evidently, with both ω_1 and ω_2 the set of nodes of the resulting graph after the processing of p and q is $V \setminus (R(p) \cup R(q))$.

We prove first that the resulting set of elements K_* is identical with both ω_1 and ω_2. Due to symmetry between ω_1 and ω_2, it suffices to show that if an element k is degraded or removed with ω_1, then it is degraded or removed with ω_2 correspondingly. Notice that, owing to the fact that p, q are dominant, one has $D(k) \cap R(p) = \emptyset = D(k) \cap R(q)$. We distinguish 3 cases:

(i) If k is removed (with ω_1) in one of the 2 reduction steps, then necessarily $R(k) \subseteq R(p) \cup R(q)$. Then k is removed also with ω_2 as well, in one of the two reduction steps.

Assume now that k is not removed. Then

(ii) If k is degraded (with ω_1) in the degradation step for p, then there exists a non-trivial path π in G between nodes of $D(k)$ through $D(k) \cup (R(p) \setminus R(k))$. Supposing that π is through $D(k) \cup R(q)$ the element k is also degraded (with ω_2) in the degradation step for q. Otherwise π contains a node from $R(p) \setminus (R(k) \cup R(q))$, and during the restriction to $V \setminus R(q)$ (with ω_2) it is shortened to a non-trivial path in $G^{V\setminus R(q)}$ between nodes of $D(k) = D(k^{V\setminus R(q)})$ through $D(k) \cup R(p) \setminus (R(k) \cup R(q)) = D(k^{V\setminus R(q)}) \cup R(p^{V\setminus R(q)}) \setminus R(k^{V\setminus R(q)})$. Therefore $k^{V\setminus R(q)}$ is degraded (with ω_2) in the degradation step for $p^{V\setminus R(q)}$.

(iii) If k is degraded (with ω_1) in the degradation step for $q^{V\setminus R(p)}$, then there exists a non-trivial path π in $G^{V\setminus R(p)}$ between nodes of $D(k)$ through $D(k) \cup R(q) \setminus (R(k) \cup R(p))$. Since for every edge (u, v) of π one has $\{u, v\} \cap R(q) \neq \emptyset$, one can use Lemma 4.4 (for $r = p$) to show that π is a path in G. Thus, π is a path in G between nodes of $D(k)$ through $D(k) \cup (R(q) \setminus R(k))$ and the element k is degraded in the degradation step for q with ω_2.

So, an element $k \in K$ is removed with ω_1 iff it is removed with ω_2 and similarly for degradation. But, if k is not removed, then k or its degraded version is restricted to $V \setminus (R(p) \cup R(q))$. Therefore, the obtained set of elements K_* is the same with both ω_1 and ω_2.

In the second part of the proof, we will verify that the resulting undirected graph G_* over $V \setminus (R(p) \cup R(q))$ is the same with both ω_1 and ω_2. Thus, suppose that $u, v \in V \setminus (R(p) \cup R(q))$, $u \neq v$. We distinguish between 4 cases (which are symmetric with respect to p and q) and show, say with ω_1, that in two of the cases, necessarily (u, v) is an edge in G_* and in remaining two cases, necessarily (u, v) is not an edge in G_*. Since the cases are symmetric with respect to p and q, the same conclusions will be obtained with ω_2.

Set $\mathcal{E} = \{l \in K;\ D(l) = \{u, v\}$ and $R(l) \subseteq R(p) \cup R(q)\}$. Let $\mathcal{P}$ denote the collection of all paths in G between u and v through $\{u, v\} \cup R(p) \cup R(q)$. The 4 cases are considered below.

(a) Assume that $\mathcal{P} = \emptyset$. Then (u, v) is not an edge in G_*.

Indeed (with ω_1): Suppose for a contradicition that (u, v) is an edge in G_*. Since (u, v) is not an edge in G, it must have been created as an edge in one of the restriction steps. Owing to the assumption ($\mathcal{P} = \emptyset$) it could not have been created (with ω_1) during restriction for p. Thus, it was created during restriction for $\tilde{q} = q^{V \setminus R(p)}$. Thus, before its creation there was a path π (in the corresponding undirected graph) between u and v through $\{u, v\} \cup R(\tilde{q})$. The same path π occurs evidently after restriction for p. Hence, before restriction for p there was a path π' between u and v through $\{u, v\} \cup R(p) \cup R(q)$. This path π' evidently occurs before degradation for p contradicting our assumption that $\mathcal{P} = \emptyset$.

(b) Assume that $\mathcal{P} \neq \emptyset$ and $\mathcal{E} = \emptyset$. Then (u, v) is an edge in G_*.

Indeed (with ω_1): Since $\mathcal{P} \neq \emptyset$ before degradation for p there was a path π between u and v through $\{u, v\} \cup R(p) \cup R(q)$. This path exists also after the degradation and during restriction for p it is shortened to a path π' between u and v through $\{u, v\} \cup R(\tilde{q})$ where $\tilde{q} = q^{V \setminus R(p)}$. This path π' could be disconnected during reduction (for p) only if one of its edges is a domain of an element $s \in K$ with $R(s) \subseteq R(p)$. This is impossible if π' contains a node of $R(\tilde{q})$ (otherwise s dominates q). Moreover, if π' consists of the edge (u, v) then this is also impossible since we assume that $\mathcal{E} = \emptyset$. Thus, π' exists also after degradation for $\tilde{q}$ and during restriction for $\tilde{q}$ it is shortened to the edge (u, v). The edge (u, v) cannot be recancelled during reduction (for $\tilde{q}$) since otherwise one derives that before degradation for p there was an element $s \in K$ with $D(s) = \{u, v\}$ and $R(s) \subseteq R(p) \cup R(q)$ contradicting our assumption that $\mathcal{E} = \emptyset$.

In the next two cases assume that $\mathcal{E} \neq \emptyset$ and denote $T = \bigcup\{R(l); l \in \mathcal{E}\}$. Evidently, $T \subseteq R(p) \cup R(q)$ and by consecutive application of (R2) one can show that there exists $k \in \mathcal{E}$ with $R(k) = T$, and we will restrict our attention to this element k.

(c) Assume that $\mathcal{P} \neq \emptyset \neq \mathcal{E}$ and every path from $\mathcal{P}$ is through $\{u, v\} \cup T$. Then (u, v) is not an edge in G_*.

Indeed (with ω_1): We distinguish two subcases.

(c1) $T \setminus R(p) \neq \emptyset$.

Of course, k was an element in the corresponding annotated graph before degradation for p. It follows from the assumption (c) that k is not degraded during the processing of p. Thus, k is restricted in the restriction step for p to $\tilde{k} = k^{V \setminus R(p)}$. Owing to (c1) $\tilde{k}$ remains an element also after reduction (for p). To show that $\tilde{k}$ persists unchanged also after degradation for $\tilde{q} = q^{V \setminus R(p)}$ suppose, for a contradiction, that before degradation for $\tilde{q}$ there was (in the corresponding undirected graph) a non-trivial path π between u and v through $\{u, v\} \cup R(\tilde{q})$ outside $R(\tilde{k}) = T \setminus R(p)$. Evidently, π was in the graph also before reduction for p. To show that π existed also before restriction for p one can use Lemma 4.4 (with $r = p$): every edge of π hits $R(q)$. Hence, π existed in the graph also before degradation for p, contradicting our assumption (c). Thus, necessarily, $\tilde{k}$ remains a non-degraded element and during restriction for $\tilde{q}$ it is changed into an element with empty range (recall that $T \subseteq R(p) \cup R(q)$). Therefore, during reduction (for $\tilde{q}$) the possible edge $\{u, v\}$ is cancelled.

(c2) $T \subseteq R(p)$.
By the same arguments as in (c1) one derives that k remains a non-degraded element after degradation for p. The difference is that during restriction for p it is directly restricted to an void element. This implies that after reduction (for p), (u, v) is not an edge in the corresponding graph. This remains evidently true also after degradation for $\tilde{q} = q^{V \setminus R(p)}$. Suppose by contradiction, that after restriction for $\tilde{q}$ it is again an edge. That means that before the restriction there was a path π (in the corresponding graph) between u and v through $\{u, v\} \cup R(\tilde{q})$. Since (u, v) was not an edge before restriction for $\tilde{q}$, π is non-trivial. Then one can show by the same arguments as in (c1) that π was a path in the graph already before degradation for p contradicting our assumption (c) (recall that $T \subseteq R(p)$ now).

The last case is the following one.

(d) Assume that $\mathcal{P} \neq \emptyset \neq \mathcal{E}$ and there exists a path from $\mathcal{P}$ containing a node outside $\{u, v\} \cup T$. Then (u, v) is an edge in G_*.

Indeed: One can deduce using Lemma 4.4 (with $r = k$) that there exists a non-trivial path in $\mathcal{P}$ which is completely outside $R(k) = T$ (if $w_i, ..., w_j$ is a subpath of that considered path between $w_i \in R(p) \cup R(q) \setminus R(k)$ and $w_j \in V \setminus R(k)$ such that $\{w_h; i < h < j\} \subseteq R(k)$ then (w_i, w_j) is an edge in $G^{V \setminus R(k)}$ and therefore in G and the considered path can be shortened).

Thus (with ω_1) we can distinguish two subcases.

(d1) There exists a non-trivial path from $\mathcal{P}$ outside T through $\{u, v\} \cup R(p)$.
Then during degradation for p every element $l \in \mathcal{E}$ is degraded (since $R(l) \subseteq T$ for every $l \in \mathcal{E}$). On the other hand, the path mentioned in (d1) is evidently saved during degradation for p and during restriction for p is shortened to the edge (u, v). This edge can be removed only in one of the reduction steps. But this is not possible since otherwise one derives that after degradation for p there exists a non-degraded element $l \in \mathcal{E}$.

(d2) There exists a path from $\mathcal{P}$ outside T which contains a node in $R(q) \setminus R(p)$.
Then one can use the same arguments as in the beginning of (d) and show, using Lemma 4.4 (this time with $r = p$), that there exists a non-trivial path from $\mathcal{P}$ outside $T \cup R(p)$. This path π is saved during processing of p. This implies that after degradation for $\tilde{q} = q^{V \setminus R(p)}$ there is no element $\tilde{l}$ in the corresponding annotated graph with $D(\tilde{l}) = \{u, v\}$ and $R(\tilde{l}) \subseteq R(\tilde{q})$ (otherwise before degradation for p there was $l \in \mathcal{E}$ with $\tilde{l} = l^{V \setminus R(p)}$ and hence $R(\tilde{l}) \subseteq R(l) \subseteq T$, i.e., the existence of π before degradation for $\tilde{q}$ implies a contradictory conclusion, that l was degraded at the degradation step). Moreover, π is

saved also during degradation for $\tilde{q}$ and in restriction for $\tilde{q}$ it is shortened to the edge (u, v). This edge cannot be removed during reduction (for $\tilde{q}$) since otherwise after degradation for $\tilde{q}$ there exists an element $\tilde{l}$ with $D(\tilde{l}) = \{u, v\}$ and $R(\tilde{l}) \subseteq R(\tilde{q})$.

□

Theorem 4.1. *Let (G, K) be a regular annotated graph over V and $t = (X, Z, Y)$ a triplet over V. Then the resulting graph obtained by the membership algorithm (in Step 3) does not depend on the choice of the scenario made in Step 1. In particular, the result of the membership algorithm does not depend on that choice.*

Remark 4.3. Notice, however, that the above mentioned graph does depend on t, or more exactly on the set XYZ.

Proof:
It suffices to show that for every couple of scenarios ρ, σ which are suitable for t (which means that the elements whose range does not intersect XYZ precede the elements whose range intersect XYZ - see Step 1 of the membership algorithm) there exists a sequence of scenarios $\rho = \omega_1, ..., \omega_h = \sigma, h \geq 1$ for K which are suitable for t such that $\forall i = 1, ..., h-1$ scenarios ω_i and ω_{i+1} differ only in the order of two consecutive elements, that is

$$\omega_i : r_1, ..., r_m, p, q, s_1, ..., s_n$$

$$\omega_{i+1} : r_1, ..., r_m, q, p, s_1, ..., s_n$$

where $m, n \geq 0$ (if $m = 0$ then no element precedes p and q in both sequences, if $n = 0$ then no element succeeds p and q in both sequences).

Indeed, the deletion step 2 and the processing of the elements $r_1, ..., r_m$ gives the same resulting annotated graph with both ω_i and ω_{i+1}. This graph is a regular annotated graph by Lemma 4.3. The only difference between the iterations of the membership algorithm is in the order of (possible restrictions of) p and q in the corresponding scenarios. If either p or q is removed during the processing of $r_1, ..., r_m$ then these scenarios coincide. If both p and q are saved (and possibly degraded or restricted), then we can apply Lemma 4.5 to the annotated graph obtained after processing $r_1, ..., r_m$. Hence, the iterations of the membership algorithm before the (possible) processing of s_1 already coincide. Thus, the resulting annotated graph is the same with both ω_i and ω_{i+1}.

Therefore, in order to verify the claim concerning the existence of a sequence of scenarios $\omega_1, ..., \omega_h$, $h \geq 2$ suppose that

$$\sigma : \ l_1, ..., l_g, l_{g+1}, ..., l_{m+n+2}$$

$$\rho = \omega_1 : \ \tilde{l}_1, ..., \tilde{l}_g, \tilde{l}_{g+1}, ..., \tilde{l}_{m+n+2}$$

where $l_1, ..., l_g$ are elements whose range does not intersect XYZ. Thus (since both σ and ρ are suitable for t) $\{l_1, ..., l_g\} = \{\tilde{l}_1, ..., \tilde{l}_g\}$ and we can pay attention to $l_1, ..., l_g$ only.

The idea is that if two consecutive elements p and q in a scenario for K do not dominate each other, then by their mutual exchange we obtain another scenario for K. Thus, we can obtain σ from $\rho = \omega_1$ by gradual mutual exchange of consecutive elements which do not dominate each other. Supposing $\tilde{l}_1, ..., \tilde{l}_f$ in a sequence ω_i which already coincides with $l_1, ..., l_f$ in σ for some $0 \leq f \leq g-1$ we can find $f+1 \leq e \leq g$ with $\tilde{l}_e = l_{f+1}$. Suppose non-trivial case $f+1 < e$; then $\tilde{l}_e$ precedes $\tilde{l}_{f+1}, ..., \tilde{l}_{e-1}$ in σ, and none of these elements dominates $\tilde{l}_e$. Since these elements precede $\tilde{l}_e$ in ω_i, $\tilde{l}_e$ does not dominate them either. Thus, $\tilde{l}_e$ can be gradually "moved forward" to obtain a scenario $\omega_j, j \geq i$ such that $\tilde{l}_1, ..., \tilde{l}_{f+1}$ in ω_j coincides with $l_1, ..., l_{f+1}$ in σ. □

4.4. Induced Independency Model

On basis of Theorem 4.1, we are entitled to give the following definition. Given a regular annotated graph (G, K) over V the *independency model induced* by it, denoted by $I(G, K)$, consists of those triplets over V which are represented in (G, K) according to the membership algorithm.

Theorem 4.2. *Let (G, K) be a regular annotated graph over V. Then $I(G, K)$ is a graphoid over V.*

Proof:
We have to show that $I(G, K)$ satisfies the graphoid properties (0)-(5). The trivial property (0) is evident: no matter which scenario for testing a triplet $t = (\emptyset, Z, Y)$ over V is chosen, t is represented in every undirected graph over V' where $YZ \subseteq V' \subseteq V$.

To verify Intersection (5) assume that triplets $\bar{t}_1 = (X, ZY, W)$ and $\bar{t}_2 = (X, ZW, Y)$ are represented in (G, K). We must show that $\bar{t} = (X, Z, YW)$ is represented in (G, K). Since $\bar{t}_1, \bar{t}_2$ and $\bar{t}_3$ involve the same set of vertices $XYZW$, one can choose in Step 1 of the membership algorithm a scenario ω for K which is simultaneously suitable for $\bar{t}_1, \bar{t}_2$ and $\bar{t}_3$. Thus, the resulting undirected graph G' from Step 3 of the membership algorithm is the same for $\bar{t}_1, \bar{t}_2$ and $\bar{t}_3$. Since $\bar{t}_1$ and $\bar{t}_2$ are represented in G' and $I(G')$ is a graphoid, $\bar{t}_3$ is represented in G'. Therefore, $\bar{t}_3$ is represented in (G, K).

The arguments showing that Symmetry (1) and Weak Union (3) hold are the same as in the case of Intersection.

To verify Decomposition (2), assume that $\tilde{t}_1 = (X, Z, YW)$ is represented in (G, K). We must show that $\tilde{t}_2 = (X, Z, Y)$ is represented in (G, K) as well. One can simply construct a scenario for K (in which every element of K occurs just once) of the form $\omega = (\omega_1, \omega_2, \omega_3)$ where ω_1 involves $k \in K$ with $R(k) \cap XYZW = \emptyset$, ω_2 involves $l \in K$ with $R(l) \cap W \neq \emptyset$ and $R(l) \cap XYZ = \emptyset$, and ω_3 involves $s \in K$ with $R(s) \cap XYZ \neq \emptyset$ (this is possible owing to Lemma 4.1). Thus ω is a scenario which is simultaneously suitable for testing $\tilde{t}_1$ and $\tilde{t}_2$. The main difference between testing those triplets using ω is that in the deletion step 2 (of the membership algorithm) in case of testing $\tilde{t}_2$, only elements from ω_3 are removed, while in case of testing $\tilde{t}_1$ both elements from ω_2 and ω_3 are removed. However, during processing of elements from ω_1, the algorithm both in case of testing $\tilde{t}_1$ and in case of testing $\tilde{t}_2$ behaves in the same way with regard to the changes in the undirected graph and in ω_1! According to the assumption $\tilde{t}_1$ is represented in the graph G^i obtained after processing of elements from ω_1. Since $I(G^i)$ is a graphoid, $\tilde{t}_2$ is represented in G^i as well. Thus, by Observation 4.2 $\tilde{t}_2$ is represented also in the graph $G^j, j \geq i$ obtained after processing elements from ω_1 and ω_2. Hence $\tilde{t}_2$ is represented in (G, K).

To verify Contraction (4) let us assume that triplets $t_1 = (X, ZY, W)$ and $t_2 = (X, Z, Y)$ are represented in (G, K). We must show that $t_3 = (X, Z, YW)$ is represented in (G, K) as well. One can again construct a scenario $\omega = (\omega_1, \omega_2, \omega_3)$ for K as described in the preceding case (Decomposition). Let (G^m, K^m, ω^m), $m \geq 1$ be the corresponding sequence of iterations of the membership algorithm for testing t_2, where G^m is over $V^m \subseteq V$. As explained in the preceding case the assumptions imply that t_1 is represented in the graph $G^i, i \geq 1$ obtained after processing elements from ω_1 and t_2 is represented in the graph $G^j, j \geq i$ obtained after processing of elements from ω_1 and ω_2. Since $I(G^i)$ is a graphoid containing t_1, it contains the triplet $t^m = (X, YZ, W \cap V^m)$ for every $m \geq i$. Thus, owing to Observation 4.2, the triplet t^m (over V^m) is represented in G^m for every $m \geq i$.

To show that t_2 is represented in G^i it suffices to verify that whenever t_2 is not represented in G^m for $i \leq m < j$ then it is not represented in G^{m+1}. By Lemma 4.3 we already know that (G^m, K^m) is a regular annotated graph and (G^{m+1}, K^{m+1}) is obtained from it by processing of a dominant element $p \in K^m$. Moreover, we know that every $l \in K^m$ originates from ω_2 and therefore $R(l) \cap W \neq \emptyset = R(l) \cap XYZ$. The fact that t_2 is not represented in G^m means that there exists a path π in G^m from a node $x \in X$ to a node $y \in Y$ which is outside Z. The path π is evidently saved during the degradation step for p (of the membership algorithm) and in the restriction step for p it is shortened to a path π'. Suppose for a contradiction that π' is disconnected in the reduction step (for p) by removal of its edge (u, v). One can assume without loss of generality that u is closer to x in π than v, and therefore the section of π between x and u is a path in G^m outside YZ. Thus, before reduction (for p) an element $\tilde{l}$ exists such that $D(\tilde{l}) = \{u, v\}$ and $R(\tilde{l}) = \emptyset$. Hence, there exists $l \in K^m$ with $D(l) = \{u, v\}$ and $\emptyset \neq R(l) \subseteq R(p)$. Take $w \in R(l) \cap W \subseteq W \cap V^m$ and by application of (R1) to $l \in K^m$ find a path in G^m between w and u through $\{u\} \cup R(l)$. This path is evidently outside YZ and can be merged with the above mentioned section of π to get a path in G^m between $x \in X$ and $w \in W \cap V^m$ which is outside YZ. This contradicts the fact that the triplet t^m is represented in G^m. Therefore, the path π' cannot be disconnected in the reduction step (for p) and there exists a path in G^{m+1} between $x \in X$ and $y \in Y$ which is outside Z. Thus, t_2 is not represented in G^{m+1}.

Therefore, both t_1 and t_2 are represented in G^i. Since $I(G^i)$ is a graphoid, t_3 is represented in G^i. Hence, t_3 is represented in (G, K). □

Definition 4.2. We say that two regular annotated graphs over the same set of vertices are *equivalent* if their induced graphoids coincide.

Observation 4.3. Let (G, K) and (G, L) be regular annotated graphs over V such that $L = K \cup \{r\}$ where r is a degraded element over V. Then $I(G, K) = I(G, L)$.

Proof:
Let (X, Z, Y) be a triplet over V and $\omega = (\omega_1, \omega_2)$ a scenario for K such that ω_1 contains all elements $l \in K$ with $R(l) \cap XYZ = \emptyset$ and ω_2 contains all elements $k \in K$ with $R(k) \cap XYZ \neq \emptyset$. One can distinguish two cases:

In case $R(r) \cap XYZ \neq \emptyset$ consider a scenario $\tilde{\omega} = (\omega_1, \omega_2, r)$ for L. Then, after the deletion step 2 the iterations of the membership algorithm based on (G, K, ω) and $(G, L, \tilde{\omega})$ coincide.

In case $R(r) \cap XYZ = \emptyset$ consider a scenario $\overline{\omega} = (\omega_1, r, \omega_2)$ for L. After the deletion step the only difference between iterations of the membership algorithm based on (G, K, ω) and $(G, L, \overline{\omega})$ is an additional degraded element $r \in L$ which is processed after K. But the processing of r is nothing but restriction to $V \setminus R(r)$ (the degradation and reduction steps are empty!). In either case, owing to Lemma 3.1 (X, Z, Y) is represented in (G, K) iff it is represented in (G, L). □

It is easy to verify that the regularity conditions (R1)-(R3) are saved during removal of a degraded element. Hence, Observation 4.3 implies the following.

Consequence 4.2. Let (G, L) be a regular annotated graph over V and $r \in L$ a degraded element. Then $(G, L \setminus \{r\})$ is an equivalent regular annotated graph. In particular, every regular annotated graph can be replaced by an equivalent regular annotated graph without degraded elements.

Remark 4.4. One can show using Lemma 4.3, Theorem 4.1 and Consequence 4.2 that the membership algorithm can be modified substantially in Step 4.1. The degradation step can be replaced by the following removal step.

4.1 *Removal:* If $s \in K$ is an element such that *there exists a non-trivial path in G between nodes of $D(s)$ through $D(s) \cup (S \setminus R(s))$*, then remove s from K and cancel it in ω.

We leave it to the reader to verify that this simplification is valid.

The case of adding a degraded element is treated in the following lemma.

Lemma 4.6. *Let (G, K) be a regular annotated graph over V and let r be a degraded element over V with $R(r) = B$ such that*

(a) $\forall k \in K$, $\forall u \in R(k)$, $\forall v \in B$, *if (u, v) is an edge in G, then $v \in R(k)$.*
(b) $\forall$ *path $w_1, \ldots, w_n$, $n \geq 3$ in G through $\{w_1, w_n\} \cup B$ such that $w_1, w_n \in V \setminus B$ and (w_1, w_n) is not K-durable edge in G, there exists $q \in K$ such that $D(q) = \{w_1, w_n\}$ and $\{w_h;\ 1 < h < n\} \subseteq R(q)$.*

Then $(G, K \cup \{r\})$ is an equivalent regular annotated graph.

Proof:
First, we need to show that the condition (R1) for (G, K) can be strengthened as follows.

(R1*) $\forall k \in K$, $\forall u \in R(k)$, $\forall v \in D(k)$, there exists a path $u = w_0, \ldots, w_m = v$, $m \geq 1$ in G through $\{v\} \cup R(k)$ composed of K-durable edges such that $K_{w_i} \subseteq K_{w_j}$ whenever $0 \leq i \leq j \leq m$ (where $K_w = \{l \in K;\ w \notin R(l)\}$ for any $w \in V$).

Indeed, let us consider $k \in K$ and $v \in D(k)$ fixed. In fact, we verify (R1*) by 'reverse' induction on the cardinality of K_u. Thus, suppose that $u \in R(k)$ is such that for every $x \in R(k)$ with $K_u \subset K_x$ was (R1*) already verified (it involves the case when u has maximal K_u within $R(k)$). By (R1) find a respective path $u = x_o, \ldots, x_s = v$ with $K_{x_0} = K_u \subseteq K_{x_i}$ for $i = 1, \ldots, s$. Find minimal $1 \leq j \leq s$ such that $K_u \neq K_{x_j}$ (observe that $k \in K_v \setminus K_u$) and put $w_i = x_i$ for $i = 0, \ldots, j$. Since $K_{w_0} = \ldots = K_{w_{j-1}} \subset K_{w_j}$ our claim follows in case $j = s$. In case $j < s$, we apply the induction hypothesis to x_j and find a path $x_j = y_0, \ldots, y_r = v$ between x_j and v satisfying the requirements of (R1*). Put $w_{j+i} = y_i$ for $i = 1, \ldots, r$. Then $w_0, \ldots, w_{j+r}$ is the required path between u and v satisfying the requirements of (R1*).

To verify (R1) for $(G, K \cup \{r\})$ consider fixed $k \in K \cup \{r\}$, $u \in R(k)$ and $v \in D(k)$. Then $k \in K$ and using (R1*) for (G, K) find the corresponding path $\pi : u = w_0, \ldots, w_m = v$, $m \geq 1$. The condition (a) implies that $v \notin B$ (one has $w_{m-1} \in R(k)$ and (w_{m-1}, v) is an edge in G). In case $u \notin B$, one can always shorten π to a path outside B satisfying (R1*). The reason is that for every $0 \leq i < j \leq m$ such that $w_i, w_j \notin B$ and $\emptyset \neq \{w_h;\ i < h < j\} \subseteq B$ necessarily (w_i, w_j) is a K-durable edge in G: otherwise the condition (b) implies the existence of $q \in K$ with $D(q) = \{w_i, w_j\}$ and $\{w_h;\ i < h < j\} \subseteq R(q)$ and the fact $q \in K_{w_i} \setminus K_{w_{i+1}}$ contradicts the condition from (R1*). The shortened path then satisfies the requirements from (R1) for $(G, K \cup \{r\})$.

The condition (R2) for $(G, K \cup \{r\})$ easily follows from (R2) for (G, K).

To verify (R3) for $(G, K \cup \{r\})$ consider $k, l \in K \cup \{r\}$ and the path $w_1, \ldots, w_n$, $n \geq 2$ mentioned in the premise of (R3). The case $k, l \in K$ is covered by (R3) for (G, K). In case $k \in K$ and $l = r$ the condition (a) implies $n = 2$ (otherwise $w_2 \in R(k)$, $w_1 \in B$ implies $w_1 \in R(k)$) in which case (R3) is trivial. In case $k = r$, $l \in K$ the required conclusion of (R3) follows from the condition (b).

Thus, $(G, K \cup \{r\})$ is a regular annotated graph and Observation 4.3 can be applied. □

5. Annotation Algorithm - Additional Proofs

In this section we continue by showing that the annotation algorithm produces a regular annotated graph, such that the graphoid represented by it is identical to the graphoid closure of the graphoids induced by the individual graphs of the input nest (the proof of the last statement is moved to Section 7). However, this section contains an illustrative example as well.

5.1. Properties, Observations and Consequences

In this section we recall, for the benefit of the reader, the basic definitions and procedures involved in the annotation algorithm and prove the properties of the resulting annotated graph in a sequence of observations, consequences and lemmas. Recall that the symbol $\tau(a, b|U||G)$ stands for the set of vertices $y \in V \setminus \{a, b\}$ such that for both $x \in \{a, b\}$, there is a path in G between x and y through $\{x\} \cup U$. The following facts follow easily from the definition.

Observation 5.1. Whenever the symbols below are defined it holds that:

(i) $\tau(a,b|U\|G) \subseteq U$,
(ii) $\tau(a,b|U\|G) \subseteq \tau(a,b|W\|G)$ whenever $U \subseteq W$,
(iii) $\tau(a,b|U\|G) \subseteq \tau(a,b|U\|F)$ whenever G is a subgraph of F,
(iv) $u \in \tau(a,b|U\|G),\ v \in U,\ (u,v)$ is an edge in $G \ \Rightarrow\ v \in \tau(a,b|U\|G)$.

To make the starting exposition of the annotation procedure clear and elegant we omitted its deeper assumptions in Definition 3.3. However, throughout this section we will use these assumptions systematically. Therefore we repeat the definition together with all relevant assumptions.

Annotation procedure. Let (H,L) be a *regular* annotated graph over V without degraded elements and without void elements. Let F be an undirected graph over $V \cup B$ (where $V \cap B = \emptyset$) such that H is a subgraph of F. We construct an annotated graph (G,K) over $V \cup B$ denoted by <u>Annot</u> $((H,L):F)$ as follows.

A1. The graph G is obtained from F by removal of those edges (u,v) in F such that $u,v \in V$, (u,v) is not an edge in H and $\tau(u,v|B\|F) = \emptyset$.

A2. Some elements of K are *newly created*: for every couple of vertices $u,v \in V$, $u \neq v$ such that (u,v) is not and edge in H and $\tau(u,v|B\|F) \neq \emptyset$ consider a non-degraded element $(\{u,v\},U)$ with $U = \tau(u,v|B\|F)$ and insert it into K (which was empty before Step A2).

A3. The other elements of K are *created by expanding* elements of L: for each $(\{u,v\},W) \in L$ incorporate into K the element $(\{u,v\},T)$ where $T = \tau(u,v|W \cup B\|F)$.

Observation 5.2. Under the assumptions of the annotation procedure one has:

(i) G is a subgraph of F,
(ii) H is a subgraph of G,
(iii) if $k \in K$ is created by expanding $l \in L$, then $D(k) = D(l)$ and $R(l) = R(k) \cap V$,
(iv) $k \in K$ is newly created iff $R(k) \subseteq B$,
(v) for every $k \in K$, $\emptyset \neq D(k) \subseteq V$ and $R(k) \neq \emptyset$.

Proof:
The fact (i) is evident from the construction. To prove (ii) notice that whenever (u,v) is an edge in H, then it is an edge in F and cannot be removed in Step A1. To verify (iii) notice that Observation 5.1 (i) implies $R(k) \subseteq R(l) \cup B$. Hence $R(k) \cap V \subseteq R(l)$. To verify $R(l) \subseteq R(k) \cap V$ use the regularity condition (R1) for (H,L): it implies $R(l) \subseteq \tau(D(l)|R(l)\|H)$. Since H is a subgraph of F, by Observation 5.1 (iii), (ii) $\tau(D(l)|R(l)|H) \subseteq \tau(D(l)|R(l)\|F) \subseteq \tau(D(l)|R(l) \cup B\|F) = R(k)$. Hence $R(l) \subseteq R(k)$. To verify (iv) observe that necessity of $R(k) \subseteq B$ is trivial by Observation 5.1 (i). The sufficiency follows from the assumption that every element of L has a non-empty range by means of (iii). The condition (v) also follows from the construction and from (iii). □

We will often utilize the following 'transitivity principle' which is a consequence of Observation 5.1 (iv) and Observation 5.2 (i).

Observation 5.3. Suppose that $(G, K) =$ Annot $((H, L) : F)$ under the assumptions of the annotation procedure. Then $\forall k \in K$, $\forall u \in R(k)$, $\forall v \in B$, if (u, v) is an edge in F or G, then $v \in R(k)$. { Notice that in this case (u, v) is an edge of F iff it is an edge of G. }

Observation 5.4. Suppose that $(G, K) =$ Annot $((H, L) : F)$ under the assumptions of the annotation procedure. Let $k \in K$ be created by expanding of $l \in L$. Then $\forall\ u \in R(k) \cap V$, $\forall v \in D(k)$, there exists a path in G between u and v through $\{v\} \cup (R(k) \cap V)$ composed of K-durable edges which is outside $R(K_u)$. In particular, $R(k) \cap V = \tau(D(k)|R(k) \cap V\|G)$.

Proof:
Let us fix $u \in R(k) \cap V = R(l)$ and $v \in D(k) = D(l)$ (see Observation 5.2 (iii)). According to (R1) for (H, L) find a path π in H between u and v through $\{v\} \cup R(l) = \{v\} \cup (R(k) \cap V)$ composed of L-durable edges and outside $R(L_u)$. By Observation 5.2 (ii) it is a path in G. Since π is a path in H the vertices of its edges cannot be domains of newly created elements of K (see Step A2 of the annotation procedure). Thus, by Observation 5.2 (iii) its L-durable edges are also K-durable. Owing to Observation 5.2 (iv) π is outside the range of all newly created elements of K. If $r \in K_u$ is created by expanding $\tilde{r} \in L$, then by Observation 5.2 (iii) $\tilde{r} \in L_u$ and π is outside $R(r)$.

The second claim in Observation 5.4 follows from Observation 5.1 (i) and the first claim. □

Observation 5.5. Suppose that $(G, K) =$ Annot $((H, L) : F)$ under the assumptions of the annotation procedure. Let l be a non-degraded element over V such that either $R(l) = \emptyset$ or $l \in L$. Then $\tau(D(l)|R(l) \cup B\|F) = \tau(D(l)|R(l) \cup B\|G)$. In particular, for $\{a, b\} \subseteq V$ we have that $\tau(a, b|B\|F) = \tau(a, b|B\|G)$.

Proof:
By Observation 5.1 (iii) and Observation 5.2 (i) $\tau(D(l)|R(l) \cup B\|G) \subseteq \tau(D(l)|R(l) \cup B\|F)$. Let us consider $u \in \tau(D(l)|R(l) \cup B\|F)$. By Observation 5.1 (i) $u \in R(l) \cup B$. In case $u \in R(l)$ use Observation 5.2 (iii) and Observation 5.4 and then Observation 5.1 (ii) to infer that $R(l) = \tau(D(l)|R(l)\|G) \subseteq (D(l)|R(l) \cup B\|G)$. In case $u \in B$ for both $v \in D(l)$ there exists a path π_v in F from u to v through $\{v\} \cup R(l) \cup B$. Let w_v be the first node of π_v outside B. Since every edge of the section of π_v between u and w_v hits B by Step A1 of the annotation procedure, this section is a path in G as well. Therefore, in the subcase $w_v \in R(l)$ one has $w_v \in \tau(D(l)|R(l) \cup B\|G)$ and repeated application of Observation 5.1 (iv) gives $u \in \tau(D(l)|R(l) \cup B\|G)$. In the subcase $w_v = v$ for both $v \in D(l)$ one gets directly $u \in \tau(D(l)|B\|G)$ and by Observation 5.1 (ii) $u \in \tau(D(l)|R(l) \cup B\|G)$.

The second claim of the lemma is a special case when $R(l) = \emptyset$. □

Consequence 5.1. Suppose that $(G, K) =$ Annot$((H, L) : F)$ under the assumptions of the annotation procedure. Then for every $k \in K$ either $R(k) \cap V = \emptyset$ or the element l over V with $D(l) = D(k)$ and $R(l) = R(k) \cap V$ belongs to L. Moreover, $R(k) = \tau(D(k)|(R(k) \cap V) \cup B\|G) = \tau(D(k)|(R(k) \cap V) \cup B\|F)$.

Proof:
In case $R(k) \cap V = \emptyset$ by Observation 5.2 (iv) k is newly created and by Step A2 of the annotation procedure $R(k) = \tau(D(k)|B\|F) = \tau(D(k)|(R(k) \cap V) \cup B\|F)$. Then by Observation 5.5 $R(k) = \tau(D(k)|B\|F) = \tau(D(k)|B\|G) = \tau(D(k)|(R(k) \cap V) \cup B\|G)$.

If $R(k) \cap V \neq \emptyset$, then by Observation 5.2 (iv) k is created by expanding $l \in L$ and Observation 5.2 (iii) can be applied. Step A3 says $R(k) = \tau(D(l)|R(l) \cup B\|F)$ and by Observation 5.5 applied to l derive $R(k) = \tau(D(l)|R(l) \cup B\|G)$. Then substitute $R(l) = R(k) \cap V$. □

Consequence 5.2. Suppose that $(G, K) = \underline{\text{Annot}}((H, L) : F)$ under the assumptions of the annotation procedure. Let $w_1, \ldots, w_n$, $n \geq 2$ be a path in G through $\{w_1, w_n\} \cup B$ such that $w_1, w_n \in V$ and either (w_1, w_n) is not an edge in H or there exists $s \in K$ with $D(s) = \{w_1, w_n\}$. Then there exists $q \in K$ such that $D(q) = \{w_1, w_n\}$ and $\{w_h;\ 1 < h < n\} \subseteq R(q)$.

Proof:
First, consider the case when (w_1, w_n) is not an edge in H. If $n = 2$, then necessarily $\tau(w_1, w_2|B\|F) \neq \emptyset$ as otherwise in Step A1 of the annotation procedure (w_1, w_2) is removed and it is not an edge in G. If $n \geq 3$, then $\emptyset \neq \{w_h;\ 1 < h < n\} \subseteq \tau(w_1, w_n|B\|G) \subseteq \tau(w_1, w_n|B\|F)$ by Observation 5.1 (iii) and Observation 5.2 (i). In either case $\tau(w_1, w_n|B\|F) \neq \emptyset$, and therefore, in Step A2 of the annotation procedure an element $q \in K$ with $D(q) = \{w_1, w_n\}$ and $\{w_h;\ 1 < h < n\} \subseteq \tau(w_1, w_n|B\|F) = R(q)$ is newly created.

Second, suppose that there exists $s \in K$ with $D(s) = \{w_1, w_n\}$. Write by Observation 5.1 (ii) and Consequence 5.1 $\{w_h;\ 1 < h < n\} \subseteq \tau(w_1, w_n|B\|G) \subseteq \tau(w_1, w_n|(R(s) \cap V) \cup B\|G) = R(s)$. □

Lemma 5.1. *Let (H, L) be a regular annotated graph over V without degraded and void elements, and F be an undirected graph over $V \cup B$ such that H is a subgraph of F. Then $(G, K) = \underline{\text{Annot}}((H, L) : F)$ is a regular annotated graph without degraded and void elements.*

Proof:
Owing to Observation 5.2 (v) one has to show that (G, K) is a regular annotated graph.

To verify (R1) for (G, K) consider a fixed $k \in K$, $u \in R(k)$, $v \in D(k)$. One can distinguish two cases. In case $u \in R(k) \cap V$ by Observation 5.2 (iv), k is created by expanding and the desired conclusion can be derived by means of Observation 5.4. In case $u \in R(k) \cap B$ by Consequence 5.1 there exists a path in G between u and v through $\{v\} \cup (R(k) \cap V) \cup B$. Let w be the first node of the path (on the way from u to v) which is outside B and ρ its section between u and w. Since every edge of ρ hits B Observation 5.2 (v) implies that everyone of its edges is K-durable. Since $u \in R(k)$, Observation 5.3 enables us to show that every node of ρ with the possible exception of w belongs to $R(k)$. Moreover, Observation 5.3 also implies for every $l \in K$ and every node x of ρ that $x \in R(l)$ implies $u \in R(l)$. Therefore ρ is outside $R(l)$ for every $l \in K_u$ and $K_u \subseteq K_w$. Altogether, ρ is a path in G between u and w through $\{w\} \cup R(k)$ composed of K-durable edges and outside $R(K_u)$. If $w = v$ we have finished. Otherwise $w \in R(k) \cap V$ and by Observation 5.4 there exists a path π in G between w and v through $\{v\} \cup (R(k) \cap V)$ composed of K-durable edges and outside $R(K_w)$. Since $K_u \subseteq K_w$, π is outside $R(K_u)$ and can be merged with ρ to get the desired path. Thus, (R1) for (G, K) was verified .

To verify (R2) for (G, K) consider $k, l \in K$ with $D(k) = D(l)$. In case one of them, for example l, is newly created, then by Consequence 5.1 and Observation 5.1 (ii), $R(l) = \tau(D(l)|B\|G) \subseteq \tau(D(l)|(R(k)\cap V)\cup B\|G)$ and the conclusion of (R2) is trivial. In case k, l are created by expanding $\tilde{k}, \tilde{l} \in L$ then, by (R2) for (H, L), there exists $\tilde{q} \in L$ with $D(\tilde{q}) = D(l)$ and $R(\tilde{q}) = R(\tilde{k}) \cup R(\tilde{l})$. Let $q \in K$ be created by expanding $\tilde{q}$. Then, by Consequence 5.1 and Observation 5.1 (ii), derive $R(k)\cup R(l) = \tau(D(l)|(R(k)\cap V)\cup B\|G)\cup\tau(D(l)|(R(l)\cap V)\cup B\|G) \subseteq \tau(D(l)|(R(q)\cap V)\cup B\|G) = R(q)$. To show that $R(q) \subseteq R(k)\cup R(l)$ consider $u \in R(q) = \tau(D(l)|(R(k)\cap V)\cup(R(l)\cap V)\cup B\|G)$. If $u \in V$ then, by Observation 5.1 (i) either $u \in R(k)\cap V$ or $u \in R(l)\cap V$. If $u \in B$, then for both $v \in D(l)$ there exists a path π_v in G between u and v through $\{v\} \cup (R(k) \cap V) \cup (R(l) \cap V) \cup B$. Let w_v be the first node of π_v outside B. In case $w_v = v$ for both $v \in D(l)$ one has $u \in \tau(D(l)|B\|G) \subseteq \tau(D(l)|(R(k) \cap V) \cup B\|G) = R(k)$ by Observation 5.1 (ii). In case $w_v \neq v$ for some $v \in D(l)$ either $w_v \in R(k) \cap V$ or $w_v \in R(l) \cap V$. Hence, by Observation 5.3 the conclusion $u \in R(k)$ or $u \in R(l)$ can be derived. Thus, $R(q) = R(k) \cup R(l)$ and (R2) for (G, K) was verified.

To verify (R3) for (G, K) let us consider $k, l \in K$ and a path $\pi : w_1, \ldots, w_n$, $n \geq 2$ in G satisfying the conditions of the premise of (R3). Suppose that π is non-trivial since otherwise the conclusion of (R3) is evident. Then $w_1, w_n \in V$ since otherwise, by Observation 5.3, a contradictory conclusion $\{w_1, w_n\}\cap R(k) \neq \emptyset$ can be derived. Since $w_1 \in R(l)\cap V$, by Observation 5.2 (iv), l is created by expanding $\tilde{l} \in L$. First, let us consider an edge $(w_i, w_{i+1}) = (u, v)$ of π with $u, v \in V$. Suppose for a while that it is not an edge in H. Then $\tau(u, v|B\|F) \neq \emptyset$ as otherwise (u, v) is not an edge in G owing to Step A1 of the annotation procedure. Thus, in Step A2 of the annotation procedure an element $q \in K$ with $D(q) = \{u, v\} = \{w_i, w_{i+1}\}$ is created and the desired conclusion of (R3) for (G, K) holds. Thus, we can assume without loss of generality that every edge of π belonging to V is an edge in H as well. We can also assume that every such an edge is K-durable as otherwise the conclusion of (R3) already holds. Second, suppose for a while that $1 \leq i < j \leq n$ are such that $w_i, w_j \in V$ and $\{w_h;\ i < h < j\} \subseteq B$. In case $\{w_i, w_j\}$ is not an edge in H or in case there exists $s \in K$ with $D(s) = \{w_i, w_j\}$ Consequence 5.2 can be applied to derive the desired conclusion of (R3) for (G, K) directly. Otherwise π can be shortened to a path π' in H (and by Observation 5.2 (ii) also in G) all whose edges are K-durable. The assumption that $\{w_1, w_n\}$ is not a K-durable edge in G implies that π' is non-trivial. Hence, $R(k) \cap V \neq \emptyset$ and by Observation 5.2 (iv) k is created by extending $\tilde{k} \in L$. The assumption that (w_1, w_n) is not a K-durable edge in G means that either (w_1, w_n) is not an edge in G or there exists $r \in K$ with $D(r) = \{w_1, w_n\}$. The former case implies by Observation 5.2 (ii) that (w_1, w_n) is not an edge in H. In the latter case either r is newly created and (w_1, w_n) is not an edge in H again (see Step A2 of the annotation procedure), or there exists $\tilde{r} \in L$ with $D(\tilde{r}) = \{w_1, w_n\}$. In either case (w_1, w_n) is not an L-durable edge in H. Thus, the conditions assumed in the premise of (R3) for (H, L) with respect to $\tilde{k}, \tilde{l} \in L$ and π' are fulfilled. Hence, there exists $\tilde{q} \in L$ satisfying the conclusion of (R3) for (H, L). It is extented to $q \in K$. Notice that whenever $1 \leq i < j \leq n$, $w_i, w_j \in V$ and $T_{ij} = \{w_h;\ i < h < j\} \subseteq B$, then $\{w_i, w_j\} \cap R(q) \neq \emptyset$ implies by Observation 5.3 that $T_{ij} \subseteq R(q)$. Thus, one can show that q satisfies the conclusion of (R3) for (G, K).

In either case, (R3) for (G, K) was verified. Hence, (G, K) is a regular annotated graph. □

Lemma 5.2. *Let (H, L) be a regular annotated graph over V without degraded and void elements, let F be an undirected graph over $V \cup B$ such that H is a subgraph of F. Put $(G, K) = \underline{\text{Annot}}((H, L) : F)$. Then a triplet over V is represented in (G, K) if and only if it is represented in (H, L). In particular, $I(H, L) \subseteq I(G, K)$.*

Proof:
Let us consider the annotated graph $(G, K \cup \{r\})$ where r is a degraded element over $V \cup B$ with $R(r) = B$. By Lemma 5.1 (G, K) is a regular annotated graph. The condition (a) in Lemma 4.6 then follows from Observation 5.3. The condition (b) in Lemma 4.6 follows from Consequence 5.2: if $\{w_1, w_n\}$ is not a K-durable edge in G, then either there exists $s \in K$ with $D(s) = \{w_1, w_n\}$ or $\{w_1, w_n\}$ is not an edge in G in which case it is not an edge in H by Observation 5.2 (ii). Thus, by Lemma 4.6 derive $I(G, K) = I(G, K \cup \{r\})$.

Suppose that a triplet (X, Z, Y) over V is tested by the membership algorithm. One can consider a scenario ω for $(G, K \cup \{r\})$ of the form $\omega = (r, \omega_1)$ where ω_1 is a scenario for (G, K) suitable for testing (X, Z, Y). Since $R(r) = B$, r cannot be dominated and therefore it is dominant. Then, during processing of the element r, in the degradation step 4.1 no other element is degraded. This follows from Consequence 5.1. The range of newly created element is in B and they do not satisfy the degrading condition for $R(r) = B$. If the domain of an expanded element is connected to a vertex in B then that vertex will be in the element's expanded range so the degrading condition does not hold for it. In the restriction step 4.2, G is restricted to G^V. In the reduction step 4.3 just the elements $q \in K$ with $R(q) \subseteq B$ (that is exactly the newly created elements of K - see Observation 5.2 (iv)) are removed and their domains are removed from G^V. It follows from the annotation procedure and Observation 5.2 (ii) (i) that (u, v) is an edge in H if and only if it is an edge in G^V and there is no newly created $q \in K$ with $D(q) = \{u, v\}$. Thus, the resulting graph after processing r is just H. By Observation 5.2 (iii) elements created by expansion are restricted to the original elements in L. Therefore (X, Z, Y) is represented in $(G, K \cup \{r\})$ iff it is represented in (H, L). □

5.2. Basic Result about the Annotation Algorithm

The input of the annotation algorithm is a *nest* of undirected graphs, that is a sequence $F_1 \ldots, F_n$, $n \geq 1$ of undirected graphs such that F_i is a subgraph of F_{i+1} for $i = 1, \ldots, n-1$. Let us denote by V_i the set of nodes of F_i and put $B_i = V_i \setminus V_{i-1}$ for $i = 1, \ldots, n$ (by definition $V_0 = \emptyset$).

The first iteration of the annotation algorithm is an annotated graph (G_1, K_1) where $G_1 = F_1$ and $K_1 = \emptyset$. Evidently, it is a regular annotated graph (without degraded and void elements) and G_1 is a subgraph of F_2.

The next iterations of the annotation algorithm are defined by induction: we put $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$ for $i = 2, \ldots, n$. The assumptions of the annotation procedure will be always fulfilled: by Lemma 5.1 (G_i, K_i) is a regular annotated graph without degraded and void elements, and by Observation 5.2 (i), G_i is a subgraph of F_i, and therefore a subgraph of F_{i+1}, for $i = 2, \ldots, n-1$. Thus, we have the following result.

Theorem 5.1. *Suppose that $F_1 \ldots, F_n,\ n \geq 1$ is a nest of undirected graphs and $(G_i, K_i),\ i = 1, \ldots, n$ the sequence of iterations of the corresponding annotation algorithm. Then, for $i = 1, \ldots, n$, (G_i, K_i) is a regular annotated graph without degraded elements and without void elements.*

Remark 5.1. There are regular annotated graphs which cannot be obtained as a result of the annotation algorithm. For example, every annotated graph (G, K) produced by this algorithm satisfies the following condition (strengthening of (R2)):

(R2*) $\forall k, l \in K$ with $D(k) = D(l)$, either $R(k) \subseteq R(l)$ or $R(l) \subseteq R(k)$.

Observation 5.6. Under the assumptions of the annotation algorithm $\forall 1 \leq i \leq j \leq n, K_i$ is obtained from K_j by 'restriction' to V_i and removal of void elements, i.e. K_i consists of those elements $\tilde{l}$ over V_i such that there exists $l \in K_j$ with $D(\tilde{l}) = D(l) \subseteq V_i$ and $R(\tilde{l}) = R(l) \cap V_i \neq \emptyset$. In particular, $\forall j = 1, \ldots, n\,,\ \forall k \in K_j\,,\ \ R(k) \cap V_1 = \emptyset$.

Proof:
The claim is trivial when $i = j$. In case $j - i = 1$ one has $(G_j, K_j) = \underline{\text{Annot}}((G_i, K_i) : F_j)$ and by Observation 5.2 (iv) every newly created element $k \in K_j = K_{i+1}$ satisfies $R(k) \cap V_i = \emptyset$. Thus, the statement for $j - i = 1$ follows from Observation 5.2 (iii)(v). The case $j - i \geq 2$ can be derived by induction on $j - i$ since the above mentioned operations of restrictions and removal are transitive. □

Lemma 5.3. *Suppose that $F_1, \ldots, F_n,\ n \geq 1$ is a nest of undirected graphs and $(G_i, K_i)\ i = 1, \ldots, n$ are iterations of the corresponding annotation algorithm. Consider $S \subseteq V_n$ such that $S = \emptyset$ or $S = R(k)$ for some $k \in K_n$. If there exists $1 \leq m \leq n$ and $c, d \in V_m \setminus S,\ c \neq d$ such that*

(a) $(c, V_m \setminus Scd, d) \in I(F_m)$,
(b) $\forall i = m+1, \ldots, n \quad \tau(c, d|B_i||F_i) \subseteq S$,

then the triplet $(c, V_n \setminus Scd, d)$ belongs to $gr\,(I(F_m) \cup \ldots \cup I(F_n))$.

Proof:
First, let us notice that S satisfies the following transitivity principle: $\forall i = 2, \ldots, n$ if $u \in S \cap V_i$, $v \in B_i$ and (u, v) is an edge in F_i, then $v \in S$. Indeed, since $S \neq \emptyset$ in this case there exists $k \in K_n$ with $S = R(k)$. By Observation 5.6 there exists $\tilde{k} \in K_i$ with $R(\tilde{k}) = S \cap V_i$. Then Observation 5.3 applied to $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$ implies that $v \in R(\tilde{k}) = S \cap V_i$.

Let us show by induction on $i = m, \ldots, n$ that $t_i \equiv (c, V_i \setminus Scd, d) \in gr\,(I(F_m) \cup \ldots \cup I(F_i))$. For $i = m$ it follows from (a). Let us fix $m < i \leq n$ and put $M = gr\,(I(F_m) \cup \ldots \cup I(F_i))$. We must show that the induction assumption $t_{i-1} \in gr(I(F_m) \cup \ldots \cup I(F_{i-1})) \subseteq M$ implies that $t_i \in M$. Define C as the set of vertices $u \in B_i \setminus S$ such that there exists a path in F_i between c and u through $\{c\} \cup (B_i \setminus S)$ and put $A = V_{i-1} \setminus Scd$, $D = B_i \setminus SC$. Then C satisfies the following transitivity principle: if $u \in cC$, $v \in B_i \setminus S$ and (u, v) is an edge in F_i, then $v \in C$.

We show first that $\tilde{t} = (c, Ad, D) \in I(F_i)$. Suppose for a contraction that there exists a path in F_i between c and $v \in D$ through $\{c, v\} \cup C \cup (S \cap V_i)$. Thus, $v \in B_i \setminus S$ and there exists $u \in cC \cup (S \cap V_i)$ such that (u, v) is an edge in F_i. If $u \in cC$, then the transitivity principle for C implies $v \in C$ which contradicts the fact $v \in D$. If $u \in (S \cap V_i)$, then the transitivity principle for S implies $v \in S$ which also contradicts the fact $v \in D$. In either case we have shown $\tilde{t} \in M$.

We verify now that $\bar{t} = (d, ADc, C) \in I(F_i)$. Suppose for a contradiction that there exists a path in F_i between d and $v \in C$ through $\{d, v\} \cup (S \cap V_i)$. Thus, there exists $u \in \{d\} \cup (S \cap V_i)$ such that (u, v) is an edge in F_i. If $u \in S \cap V_i$, then the transitivity principle for S implies $v \in S$ which contradicts the fact $v \in C$. If $u = d$, then the fact that $v \in C$ implies that $v \in \tau(c, d|B_i\|F_i)$ by definition, and the condition (b) implies $v \in S$ which again contradicts the fact that $v \in C$. In either case, we have shown that $\bar{t} \in M$.

Since M is a graphoid, the facts $t_{i-1} = (c, A, d) \in M$ and $\tilde{t} = (c, Ad, D) \in M$ imply by Contraction (4) that $(c, A, Dd) \in M$. Hence by Weak Union (3) $(c, AD, d) \in M$ and by Symmetry (1) $(d, AD, c) \in M$. This together with the fact $\bar{t} = (d, ADc, C) \in M$ implies by Contraction that $(d, AD, cC) \in M$. Hence, by Weak Union $(d, ADC, c) \in M$ and by Symmetry $t_i = (c, ADC, d) \in M$. This concludes the induction step. □

Definition 5.1. We say that a nest of undirected graphs $F_1, \ldots, F_n$, $n \geq 1$ is *regular* if

$$\forall i = 2, \ldots, n\,,\ \forall u, v \in V_{i-1}\,,\ \text{ if } (u, v) \text{ is an edge in } F_i \text{ but not in } F_{i-1}, \text{ then } \tau(u, v|B_i\|F_i) \neq \emptyset\,.$$

We leave it to the reader to verify that this is a necessary and sufficient condition for a nest of undirected graphs not to be modified during the annotation algorithm, i.e. $F_i = G_i$ for $i = 1, \ldots, n$. The following lemma says that we can limit our attention to regular nests.

Lemma 5.4. *Let $F_1, \ldots, F_n$, $n \geq 1$ be a nest of undirected graphs and (G_i, K_i), $i = 1, \ldots, n$ the sequence of graphs generated by the corresponding annotation algorithm. Then $G_1, \ldots, G_n$ is a regular nest of undirected graphs, moreover the annotation algorithm applied to $G_1, \ldots, G_n$ results in the same sequence of iterations and*

$$\forall i = 1, \ldots, n \quad gr(I(G_1) \cup \ldots \cup I(G_i)) = gr(I(F_1) \cup \ldots \cup I(F_i)).$$

Proof:
Observation 5.2 (ii) implies that G_{i-1} is a subgraph of G_i for $i = 2, \ldots, n$. To show that $G_1, \ldots, G_n$ is a regular nest, suppose for a contradiction that there exists $i \in \{2, \ldots, n\}$ and $\{u, v\} \in V_{i-1}$ such that (u, v) is an edge in G_i but not in G_{i-1} and $\tau(u, v|B_i\|G_i) = \emptyset$. Then by Observation 5.5 $\tau(u, v|B_i\|F_i) = \tau(u, v|B_i\|G_i) = \emptyset$. Thus, in Step A1 of the annotation procedure $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$ the edge (u, v) in F_i is removed which contradicts the fact that it is an edge in G_i.

Now put $G'_1 = G_1$, $K'_1 = \emptyset$ and define by induction $(G'_i, K'_i) = \underline{\text{Annot}}((G'_{i-1}, K'_{i-1}) : G_i)$ for $i = 2, \ldots, n$. It suffices to show by induction that $(G'_i, K'_i) = (G_i, K_i)$ for $i = 1, \ldots, n$. By definition, this is true for $i = 1$. Let us consider $1 < i \leq n$. Then in Step A1 of the annotation procedure $(G'_i, K'_i) = \underline{\text{Annot}}((G'_{i-1}, K'_{i-1}) : G_i)$ owing to the induction assumption $G'_{i-1} = G_{i-1}$ and, by the regularity condition for $G_1, \ldots, G_n$, no edge is removed and therefore $G'_i = G_i$. Then, in Step A2 of the annotation procedure $(G'_i, K'_i) = \underline{\text{Annot}}((G'_{i-1}, K'_{i-1}) : G_i)$, the respective assumptions $u, v \in V_{i-1}$, $u \neq v$, (u, v) is not an edge in G'_{i-1} and $\tau(u, v|B_i\|G_i) \neq \emptyset$ are equivalent to the respective assumption in Step A2 of the procedure $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$. Indeed, one has $G'_{i-1} = G_{i-1}$ and Observation 5.5 for $(G_i, K_i) = \underline{\text{Annot}}\ ((G_{i-1}, K_{i-1}) : F_i)$ implies that $\tau(u, v|B_i\|G_i) = \tau(u, v|B_i\|F_i)$. Thus, the same elements are newly created. In Step A3 for every $(\{u, v\}, L) \in K'_{i-1} = K_{i-1}$ (the induction assumption), Observation 5.5 implies $\tau(u, v|L \cup B_i\|G_i) = \tau(u, v|L \cup B_i\|F_i)$ and the element is expanded in the same way. Thus, $K'_i = K_i$ and we have shown that the annotation algorithm for $G_1, \ldots, G_n$ and $F_1, \ldots, F_n$ gives the same output.

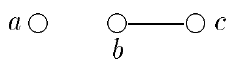
By Observation 5.2 (i) G_i is a subgraph of F_i for $i = 1, \ldots, n$. Thus $I(F_i) \subseteq I(G_i)$ and hence $gr\,(I(F_1) \cup \ldots \cup I(F_i)) \subseteq gr\,(I(G_1) \cup \ldots \cup I(G_i))$ for $i = 1, \ldots, n$. To verify the converse inclusion it suffices to show that $I(G_j) \subseteq gr\,(I(F_1) \cup \ldots \cup I(F_j))$ for $j = 1, \ldots, n$. It is evident for $j = 1$. In case $j \geq 2$ put $M = gr\,(I(F_1) \cup \ldots \cup I(F_j))$. Since M is a graphoid by Claim 2.1 applied to G_j we need to show that every triplet of the form $t = (c, V_j \setminus cd, d)$ such that $c, d \in V_j$, $c \neq d$ and (c, d) is not an edge in G_j, belongs to M. We can apply Lemma 5.3 with $n = j$ and $S = \emptyset$. Since (c, d) is not an edge in G_j, by Observation 5.2 (ii) it is not an edge in G_i for $1 \leq i \leq j$. In particular, it is not an edge in $F_1 = G_1$. Now put $m = \max\{1 \leq i \leq j;\ (c, d) \text{ is not an edge in } F_i\}$. Thus, (c, d) is not an edge in F_m and therefore $(c, V_m \setminus cd, d) \in I(F_m)$. Moreover, (c, d) is an edge in F_i for $i = m + 1, \ldots, j$. Since it is not an edge in G_i for $m + 1 \leq i \leq j$, by Step A1 of the annotation procedure $(G_i, K_i) = \underline{\text{Annot}}((G_{i-1}, K_{i-1}) : F_i)$ necessarily $\tau(c, d|B_i\|F_i) = \emptyset$. Thus, the conditions (a)(b) of Lemma 5.3 hold and $t \in gr\,(I(F_m) \cup \ldots \cup I(F_j)) \subseteq M$. The proof is complete. □

5.3. Example and Main Result

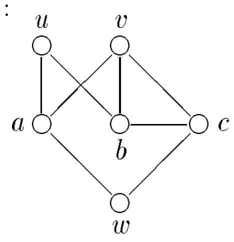
The nest of four undirected graphs $G_1, \ldots, G_4$ shown in Figure 8 is regular since every edge (u, v) in G_{i+1} with $u, v \in V_i$ is an edge in G_i, $i = 1, 2, 3$. It has been annotated according to the annotation algorithm. Here k''_i is expanded k'_i for $i = 1, 2$ and k'_i is expanded to k_i for $i = 1, 2, 3$. Notice the following:

1. k_4 and k_2 are dominant in K_4 and $k_4 \succ k_3, k_4 \succ k_1$ so $R(k_4) \subset R(k_3)$ and $R(k_4) \subset R(k_1)$. (see Lemma 4.1).
2. k_3 and k_1 have the same domain and $R(k_3) \subset R(k_1)$. The same is true for k'_3 and k'_1. It illustrates property $(R2*)$ mentioned in Remark 5.1.
3. $R(k_i) \supseteq R(k'_i) \supseteq R(k''_i)$ for $i = 1, 2$. It illustrates Observation 5.6.
4. A scenario for (G_4, K_4) must start either with k_4 or with k_2 (dominant elements). If it starts with k_2, then it must continue with k_4. If it starts with k_4, then all the possible

G_1 :

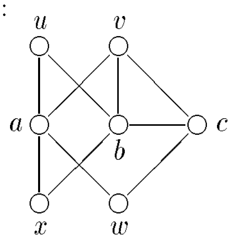


G_2 :



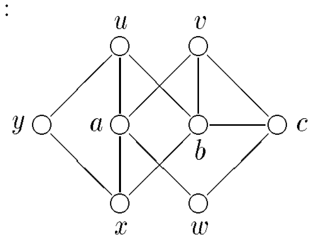
K_2 : $k''_1 = (\{a, b\}, \{u, v\})$
$k''_2 = (\{a, c\}, \{v, w\})$

G_3 :



K_3 : $k'_1 = (\{a, b\}, \{u, v, x\})$
$k'_2 = (\{a, c\}, \{v, w\})$
$k'_3 = (\{a, b\}, \{x\})$

G_4 :



K_4 : $k_1 = (\{a, b\}, \{u, v, x, y\})$
$k_2 = (\{a, c\}, \{v, w\})$
$k_3 = (\{a, b\}, \{x, y\})$
$k_4 = (\{x, u\}, \{y\})$

Figure 8. A nest of annotated graphs.

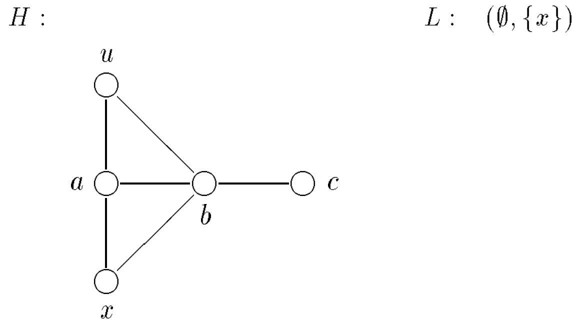


Figure 9. The second iteration of the membership algorithm.

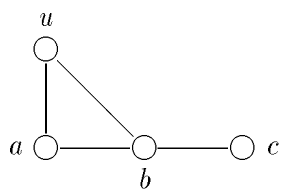


Figure 10. The last iteration of the membership algorithm.

permutations of the remaining elements are allowed. All in all, there are therefore 8 scenarios for (G_4, K_4).

Assume that we want to check whether the triplet $t = (c, bu, a)$ is represented in (G_4, K_4) according to the membership algorithm. Since $R(k_1) \cap \{a, b, c, u\} = \{u\}$ the only deleted element is k_1. We may therefore have only the following 3 scenarios after the deletion step:

$$\omega_1 = (k_4, k_3, k_2), \quad \omega_2 = (k_4, k_2, k_3), \quad \omega_3 = (k_2, k_4, k_3)\ .$$

Notice that if k_2 is processed before k_3 (as in the last 2 scenarios) the element k_3 (or its restricted version) is degraded at Step 4.1 for k_2 (or its restricted version). So, if we choose ω_2 we get, after one iteration the graph G_3 with annotation $K = \{k_2', k_3'\}$, and after two iterations, the graph (H, L) in Figure 9 (with a degraded element in L). Its processing results in the graph in Figure 10 with no element.
If we choose ω_1 then the graph in Figure 10 is also derived after 3 iterations. As it is easy to see, t is represented in this graph, therefore $t \in I(G_4, K_4)$ and $t \in gr(I(G_1) \cup \ldots \cup I(G_4))$. Indeed (c, b, a) is represented in G_1, (c, ba, u) is represented in G_2, by Contraction (4) we derive (c, b, au) and by Weak Union (3) we get (c, bu, a).

We are now ready to state the main result of our paper.

Theorem 5.2. *Suppose that $F_1, \ldots, F_n$, $n \geq 1$ is a nest of undirected graphs. Let (G_i, K_i), $i = 1, \ldots, n$ be the sequence of iterations of the corresponding annotation algorithm for this nest. Then*

$$I(G_n, K_n) = gr\,(I(G_1) \cup \ldots \cup I(G_n)) = gr\,(I(F_1) \cup \ldots \cup I(F_n))\,.$$

The proof of the above theorem is quite long and it is given after Section 6.

6. Discussion

6.1. Annotated Graphs as a Mode of Representation

Trivially, annotated graphs include UGs which correspond to the particular case where the nest of UGs consists of a single graph.

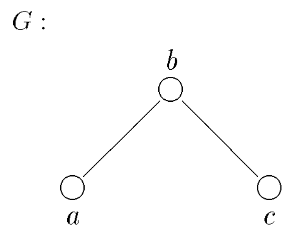
DAGs can also be represented as annotated graphs. It follows from a theorem of Pearl and Verma [6] that every DAG represents the graphoid closure of a set of triplets (stratified protocol) of the form $\{(v_i, p(v_i), a(v_i));\, 2 \leq i \leq n\}$ where $v_1, \ldots, v_n$ is a sequence including all vertices of the DAG ordered in compliance with the directionality of the edges in the DAG, $p(v_i)$ is the set of 'parent' vertices of v_i and $a(v_i)$ is the set of remaining vertices preceding v_i. Now it is easy to show that the above set of triplets (stratified protocol) can be represented as a nest of graphs.

Chain graphs (CGs) were introduced by Lauritzen and Wermuth [3] and developed by Frydenberg [1]. They generalize both DAGs and UGs as a mode of representation of irrelevance relations. Basically, they can be described as acyclic graphs whose edges may be directed or undirected. The structure of a CG is defined by the underlying undirected graph and by an ordered partition of its vertices called a *chain* $\mathcal{C} = (V(1), V(2), \cdots, V(m))$. An edge in the underlying graph which connects between a vertex in $V(i)$ and a vertex in $V(j)$ such that $i < j$ is directed from $V(i)$ to $V(j)$. The other edges in the underlying graph remain undirected. Representation in CGs is defined in a way which is similar to the definition of representation in DAGs. In [1] a 'moralization criterion' for representing triplets in a CG is described. In [8] an equivalent c-separation criterion for CGs is described which generalizes the d-separation criterion for DAGs [5]. It follows from the above mentioned papers that irrelevance relations induced by CGs can be represented as the graphoid closure of a nest of UGs. Therefore, irrelevance relations induced by CGs can be represented by annotated graphs as well. The reader is referred to above papers for a more detailed exposition (see Remark 6.1).

We will show now by an example that annotated graphs properly include the above three modes of representations. Consider the annotated graph in Figure 11. This regular annotated graph represents the set of triplets

$$\{\, (a, b, c), (a, \emptyset, c) + \text{ symmetrical images} + \text{ trivial triplets} \,\}\,.$$

The above set (which is closed under the graphoid axioms) is not closed under Transitivity (7): $(a, \emptyset, c) \not\Rightarrow (a, \emptyset, b) \vee (b, \emptyset, c)$ and therefore cannot be represented by an UG. It is not closed under Weak transitivity (9): $(a, \emptyset, c) \wedge (a, b, c) \not\Rightarrow (a, \emptyset, b) \vee (b, \emptyset, c)$ as well and therefore it cannot be



$K: \; (\{a, c\}, \{b\})$

Figure 11. Simple regular annotated graph.

represented by a DAG. The above mentioned set of triplets cannot be represented by a CG for the same reason.

As the final example consider the probability distribution given by the table below.

x	0	0	0	0	1	1	1	1
y	0	0	1	1	0	0	1	1
z	0	1	0	1	0	1	0	1
$P(\mathbf{xyz})$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$

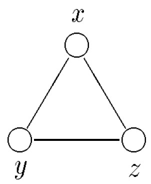
For every valuation $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ of attributes x, y, z we have that $P(\mathbf{x}) = P(\mathbf{y}) = P(\mathbf{z}) = \frac{1}{2}$ and $P(\mathbf{xy}) = P(\mathbf{xz}) = P(\mathbf{yz}) = \frac{1}{4}$, but $P(\mathbf{xyz})$ has both positive and zero values. It is therefore easy to verify that the relation induced by this probability distribution is

$$I = \{ \; (x, \emptyset, y), \; (x, \emptyset, z), \; (y, \emptyset, z) + \text{ symmetrical images } + \text{ trivial triplets } \} \, .$$

This relation cannot be represented by an UG since it does not satisfy Strong union axiom (6) (otherwise $I(x, \emptyset, y) \Rightarrow I(x, z, y)$). The relation cannot be represented by a DAG either, since a DAG over $V = \{x, y, z\}$ with an edge $a \to b$ does not represent the triplet $(a, \emptyset, b)$ and the DAG with no edge is in fact an UG. We can show in a similar way that it cannot be represented by a CG. It can be represented, however, by the annotated graph given in Figure 12. Notice that this annotated graph cannot be derived from a nest of graphs. Indeed, it is not a regular annotated graph since it does not satisfy (R1). Despite the fact that $\preceq$ is not a partial ordering on K in this case, one can formally apply the membership algorithm with an arbitrary sequence ω of all elements of K and obtain the relation induced by the above given probability distribution. This example points to the possibility of extending the results in this paper into a more general case.

Remark 6.1. The fact that chain graph representation satisfies the graphoid axioms is shown in [8]. The fact that the irrelevance relation induced by a CG is the graphoid closure of a special nest of UGs (namely the nest $G_1, \ldots, G_n$ where G_i is the moral graph of the induced graph $G_{V(1) \cup \ldots \cup V(i)}$ for $i = 1, \ldots, n$) can be derived from Consequence 3.1 of that paper.

$G:$



$K: \quad (\{x,y\},\{z\})$
$(\{x,z\},\{y\})$
$(\{y,z\},\{x\})$

Figure 12. Non-regular annotated graph.

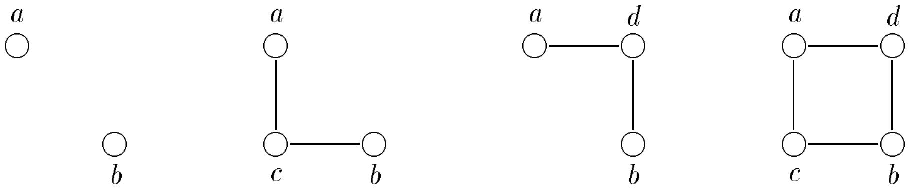
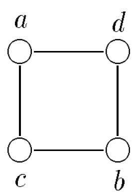


Figure 13. A lattice of UGs.

6.2. Open problems

1. Is it possible to extend the annotation algorithm or procedure so as to represent more general sets of graph (including e.g. lattices of graphs)?
   Consider, for example, the sequence of graphs given in Figure 13 which is not a nest according to our definition. The graphoid closure $gr(I(G_1) \cup \ldots \cup I(G_4))$ can be represented by the regular annotated graph from Figure 14 which cannot be obtained by means of the annotation algorithm.
2. Find a minimal set of conditions such that any given annotated graph satisfying them is the result of the annotation algorithm when applied to a nest of UGs.
3. Can two annotated graphs over the same set of vertices be combined in a meaningful way under operations induced by the graphoid axioms?

$G:$



$K: \quad (\{a,b\},\{c\})$
$(\{a,b\},\{d\})$
$(\{a,b\},\{c,d\})$

Figure 14. Regular annotated graph without (R2*) property.

4. Characterize the class of graphoids representable by annotated graphs in terms of axioms of the type described in Section 2.
5. Characterize annotated graphs that represent irrelevance relations induced by probability distributions.
6. Given a probabilistic distribution find an annotated graph which is an I-map of the distribution in a optimal way.

7. Proof of Theorem 5.2

This section contains observations and lemmas leading to the proof of Theorem 5.2.

Observation 7.1. Let (F, J) be a regular annotated graph over $\tilde{V}$ and $\tilde{S} \subseteq \tilde{V}$ such that either $\tilde{S} = R(l)$ for a dominant element $l \in J$ or $\tilde{S} = \emptyset$. Let (F^*, J^*) be derived from (F, J) by the processing of $\tilde{S}$. Then for every $u, v \in \tilde{V} \setminus \tilde{S}$, $u \neq v$; (u, v) is an edge in F^* if and only if the following two conditions hold:

[a] there exists a path in F between u and v through $\{u, v\} \cup \tilde{S}$,
[b] every $q \in J$ with $D(q) = \{u, v\}$ and $R(q) \subseteq \tilde{S}$ is degraded during Step 4.1 (for $\tilde{S}$).

Proof:
Suppose that (u, v) is an edge in F^*. It is an edge before the reduction step 4.3 as well. Therefore, before the restriction step (and hence before the degradation step) the path mentioned in [a] exists. Suppose for contradiction that the condition [b] is not valid, i.e. there exists an element $q \in J$ with $D(q) = \{u, v\}$ and $R(q) \subseteq \tilde{S}$ which is not degraded in Step 4.1 (for $\tilde{S}$). Then, in the restriction step it is changed into a non-degraded element with empty range. Thus, in the reduction step (u, v) is cancelled as an edge which contradicts the assumption that (u, v) is an edge in F^*.

Conversely, suppose that both [a] and [b] hold. Then the path mentioned in [a] exists also after the degradation step. Moreover, owing to [b], after the degradation step no element q with $D(q) = \{u, v\}$ and $R(q) \subseteq \tilde{S}$ exists. Thus, after the restriction step, (u, v) is an edge in the graph and there is no element q with $D(q) = \{u, v\}$ and $R(q) = \emptyset$. Therefore, the edge is not removed during the reduction step and (u, v) is an edge in F^*. □

7.1. Commutativity Lemma

The proof of Theorem 5.2 is based on a special 'commutativity lemma' saying that the steps of the annotation algorithm and the membership algorithm commute. The assumptions of this lemma (Lemma 7.2) are quite complex and are described below.

Assumptions and notation for the commutativity lemma.
Let (H, L) be a regular annotated graph over V without degraded element and void elements. Let F be an undirected graph over $V \cup B$ (assuming that $V \cap B = \emptyset$) such that H is a subgraph of F. Denote $(G, K) = \underline{\text{Annot}}((H, L) : F)$. Let $S \subseteq V \cup B$ be a set such that either $S = R(k)$

for a dominant element $k \in K$ or $S = \emptyset$. Denote by (G^*, K^*) the annotated graph derived from (G, K) by the processing of S followed by the removal of resulting degraded elements. Similarly, denote by (H^*, L^*) the annotated graph derived from (H, L) after the processing of $S \cap V$ and the removal of the resulting degraded elements.

Remark 7.1. Lemma 5.1 implies that (G, K) is a regular annotated graph without degraded and void elements. Consequences 4.1 and 4.2 imply that (G^*, K^*) is also a regular annotated graph without degraded and void elements. Observation 5.2 (iv) (iii) makes it possible to show that $S \cap V = R(\tilde{k})$ for a dominant element $\tilde{k} \in L$ or $S \cap V = \emptyset$. Thus, by Consequences 4.1 and 4.2, (H^*, L^*) is also a regular annotated graph without degraded and void elements.

Observation 7.2. Under assumptions of the commutativity lemma suppose that $u, v \in V \setminus S$, $u \neq v$. Then every path ρ in G between u and v through $\{u, v\} \cup S$ which hits $S \cap V$ can be shortened to a (non-trivial) path in H between u and v through $\{u, v\} \cup (S \cap V)$.

Proof:
It suffices to show for every section $c = x_1, \ldots, x_i = d$, $i \geq 2$ of ρ such that $c, d \in V$ and $x_2, \ldots, x_{i-1} \notin V$ that (c, d) is an edge in H. Suppose for a contradiction that it is not the case. Then $\tau(c, d|B||F) \neq \emptyset$. Indeed, it is trivial in case $i \geq 3$ since G is a subgraph of F. In case $i = 2$, i.e. (c, d) is an edge in G, this follows from Step A1 of the annotation procedure, since otherwise the edge (c, d) has to be removed from F during the step. The fact $T \equiv \tau(c, d|B||F) \neq \emptyset$ implies that during Step A2 of the annotation procedure, an element $q \in K$ with $D(q) = \{c, d\}$ and $R(q) = T$ is newly created. However, the assumption that ρ hits $S \cap V$ implies that $\{c, d\} \cap S \neq \emptyset$ so that q dominates k. But $S = R(k)$ for a dominant element $k \in K$ which contradicts the fact that q dominates k. □

Observation 7.3. Under assumptions of the commutativity lemma suppose that $l \in K$ is created by expanding $\tilde{l} \in L$. Then l is degraded in the degradation step of processing of S if and only if $\tilde{l}$ is degraded in the degradation step of processing of $S \cap V$.

Proof:
Suppose that $\tilde{l}$ is degraded, that is there exists a non-trivial path in H between nodes of $D(\tilde{l})$ through $D(\tilde{l}) \cup (S \cap V) \setminus R(\tilde{l})$. Owing to Observation 5.2 (ii) (iii) it is a non-trivial path in G between nodes of $D(l) = D(\tilde{l})$ through $D(l) \cup (S \cap V) \setminus (R(l) \cap V) \subseteq D(l) \cup S \setminus R(l)$. Thus, $l \in K$ is degraded in the degradation step of processing of S.

Conversely, suppose that $l \in K$ is degraded, that is there exists a non-trivial path ρ in G between nodes of $D(l)$ through $D(l) \cup S \setminus R(l)$. To show that ρ hits $S \cap V$ suppose for a contradiction that it is a path through $D(l) \cup B$. Then by Observation 5.1 (ii) and Consequence 5.1 its internal nodes belong to $\tau(D(l)|B||G) \subseteq \tau(D(l)|(R(l) \cap V) \cup B||G) = R(l)$ which contradicts the fact that ρ is outside $R(l)$. Thus, ρ has to hit $S \cap V$ and by Observation 7.2 can be shortened to a non-trivial path in H between nodes of $D(\tilde{l}) = D(l)$ through $D(\tilde{l}) \cup (S \cap V)$. Of course, the shortened path is outside $R(\tilde{l}) \subseteq R(l)$ (see Observation 5.2 (iii)). Therefore, $\tilde{l}$ is degraded in the degradation step of processing of $S \cap V$. □

Observation 7.4. Under assumptions of the commutativity lemma suppose that $l \in K$ is newly created in the annotation procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$. Then l is degraded in the degradation step of processing of S if and only if there exists a path in H between nodes of $D(l)$ through $D(l) \cup (S \cap V)$.

Proof:
We know by Observation 5.2 (v) that $D(l) \subseteq V$. It follows from Step A2 of the annotation procedure that $D(l)$ is not an edge in H. Therefore, if there exists a path in H between nodes of $D(l)$ through $D(l) \cup (S \cap V)$, then it is a non-trivial path. By Observation 5.2 (ii) (iv) it is a non-trivial path in G between nodes of $D(l)$ through $D(l) \cup (S \cap V) \subseteq D(l) \cup S \setminus R(l)$. Thus, $l \in K$ is degraded in the degradation step of processing of S.

Conversely, suppose that l is degraded, that is there exists a non-trivial path ρ in G between nodes of $D(l)$ through $D(l) \cup S \setminus R(l)$. Then ρ has to hit $S \cap V$ as otherwise by Consequence 5.1 its internal nodes belong to $\tau(D(l)|B\|G) = R(l)$ which contradicts the assumption. By Observation 7.2 ρ can be shortened to the desired path in H. □

Lemma 7.1. *Under assumptions of the commutativity lemma H^* is a subgraph of G^*.*

Proof:
Suppose that $u, v \in V \setminus S$ and (u, v) is an edge in H^*. According to Observation 7.1 where $(F, J) = (H, L)$, $\tilde{V} = V$ and $\tilde{S} = S \cap V$ the following two conditions hold.

(a) there exists a path in H between u and v through $\{u, v\} \cup (S \cap V)$,

(b) every $\tilde{l} \in L$ with $D(\tilde{l}) = \{u, v\}$ and $R(\tilde{l}) \subseteq S \cap V$ is degraded during Step 4.1 for $S \cap V$.

Owing to Observation 5.2 (ii) the condition (a) implies:

(a′) there exists a path in G between u and v through $\{u, v\} \cup S$.

Moreover, the following condition holds:

(b′) every $l \in K$ with $D(l) = \{u, v\}$, $R(l) \subseteq S$ is degraded during the degradation step for S.

Indeed, in case $R(l) \cap V = \emptyset$, l is newly created in the procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$ by Observation 5.2 (iv), and the condition (a) implies according to Observation 7.4 that l is degraded. In case $R(l) \cap V \neq \emptyset$, l is created by the expansion of $\tilde{l} \in L$ with $D(\tilde{l}) = \{u, v\}$ and $R(\tilde{l}) \subseteq S \cap V$ (see Observation 5.2 (iii)). Then the condition (b) implies by Observation 7.3 that l is degraded and (b′) is verified. It remains to use Observation 7.1 where $(F, J) = (G, K)$, $\tilde{V} = V \cup B$ and $\tilde{S} = S$ to show that the conditions (a′), (b′) imply that $\{u, v\}$ is an edge in G^*. □

Observation 7.5. Under the assumptions of the commutativity lemma $\forall l \in L$, $\forall u \in R(l) \setminus S$, $\forall v \in D(l)$, there exists a path simultaneously in G and G^* from u to v through $\{v\} \cup R(l) \setminus S$.

Proof:
It follows from the annotation procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$ that there exists $k \in K$ created by the expanding of $l \in L$ and by Observation 5.2 (iii) $R(l) = R(k) \cap V$. Let us apply Observation 5.4 to $k \in K$, $u \in R(l) \setminus S = (R(k) \cap V) \setminus S$ and $v \in D(l) = D(k)$ to find a path ρ in G between u and v through $\{u\} \cup (R(k) \cap V)$ composed of K-durable edges and outside $R(K_u)$. If $S \cap V = \emptyset$, then ρ is evidently outside S. In case $S \cap V \neq \emptyset$ consider $p \in K$ with $R(p) = S$. Since $u \notin R(p)$, ρ is outside $R(p) = S$. Thus, the path ρ will remain in the graph after the degradation and restriction steps of processing of S. Moreover, since it is made of K-durable edges it cannot be disconnected in the reduction step of processing of S. Therefore, ρ is also a path in G^*. □

Observation 7.6. Under assumptions of the commutativity lemma suppose that l is a non-degraded element over V such that either $l \in L$ or $D(l) \cap S = \emptyset = R(l)$. Then one has $\tau(D(l)|R(l) \cup B\|G) \setminus S = \tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)\|G^*)$.

Proof:
First, it follows from the assumptions and Observation 5.3 that S satisfies the following transitivity principle: if $x \in S$, $y \in B$ and (x, y) is an edge in G, then $y \in S$.

Let us suppose that $u \in \tau(D(l)|R(l) \cup B\|G) \setminus S$. Then by Observation 5.1 (i) $u \in (R(l) \cup B) \setminus S$. Let us consider a fixed $v \in D(l)$. Then, by definition, there exists a path ρ in G from u to v through $\{v\} \cup R(l) \cup B$. In case $u \in R(l) \setminus S$ the existence of the desired path in G^* from u to v through $\{v\} \cup R(l) \setminus S \subseteq \{v\} \cup (R(l) \setminus S) \cup (B \setminus S)$ follows from Observation 7.5. In case $u \in B \setminus S$ denote by x_v the first node of ρ outside B. Then the section of ρ between u and x_v is outside S owing to the above mentioned transitivity principle for S. Since all its edges intersect B, by Observation 5.2 (v) all its edges are K-durable. Hence, the section of ρ between u and x_v remains unchanged during processing of S. So, it is a path in G^*. Thus, in case $x_v = v$ for both $v \in D(l)$ the statement $u \in \tau(D(l)|B \setminus S\|G^*) \subseteq \tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)\|G^*)$ is verified. If $x_v \neq v$ for some $v \in D(l)$, then $x_v \in R(l) \setminus S$ and we already know that $x_v \in \tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)\|G^*)$. Then by repeated application of Observation 5.1 (iv) derive that every node of the section of ρ between x_v and u belongs to $\tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)\|G^*)$.

Conversely, suppose that $u \in \tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)\|G^*)$. Then by Observation 5.1 (i), $u \in (R(l) \cup B) \setminus S$. Let us consider a fixed $v \in D(l)$. By definition, there exists a path π in G^* from u to v through $\{v\} \cup (R(l) \setminus S) \cup (B \setminus S)$. In case $u \in R(l) \setminus S$ the existence of the desired path in G from u to v through $\{v\} \cup R(l) \setminus S \subseteq \{v\} \cup R(l) \cup B$ follows from Observation 7.5. If $u \in B \setminus S$ denote by w_v the first node of π outside B. Evidently, every edge of the section of π between u and w_v is an edge of the graph before the reduction step of processing of S. To show that every edge $\{z, w\}$ of the section is an edge in the graph also before the restriction step of processing of S, suppose for a contradiction that there exists a non-trivial path in G between z and w through $\{z, w\} \cup S$. Then, the fact $\{z, w\} \cap B \neq \emptyset$ implies by the above mentioned transitivity principle $\{z, w\} \cap S \neq \emptyset$ which contradicts the fact that π is outside S. Thus, the section of π between u and w_v exists in the graph before processing of S, that is, it is a path in G. If $w_v = v$ for both $v \in D(l)$, then it says $u \in \tau(D(l)|B \setminus S\|G) \subseteq \tau(D(l)|R(l) \cup B\|G) \setminus S$. If $w_v \neq v$

for some $v \in D(l)$, then $w_v \in \tau(D(l)|R(l) \cup B\|G)$ and repeated application of Observation 5.1 (iv) implies that $u \in \tau(D(l)|R(l) \cup B\|G)$. □

The desired commutativity lemma follows.

Lemma 7.2. *Under assumption of the commutativity lemma* $(G^*, K^*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$.

Proof:
By Consequences 4.1 and 4.2 (H^*, L^*) is a regular annotated graph over $V \setminus S$ without degraded and void elements. By Lemma 7.1 G^* is an undirected graph over $(V \setminus S) \cup (B \setminus S)$ such that H^* is a subgraph of G^*. Thus, the assumptions of the annotation procedure are fulfilled and we can introduce $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$.

To show that $G_* = G^*$ it suffices to verify that in Step A1 of the annotation procedure $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ no edge is removed. That is, whenever $u, v \in V \setminus S$ such that (u, v) is an edge in G^* but not in H^*, then we must show that $\tau(u, v|B \setminus S\|G^*) \neq \emptyset$. Since it is not an edge in H^* by Observation 7.1 where $(F, J) = (H, L)$, $\tilde{V} = V$ and $\tilde{S} = S \cap V$ derive that one of the following two conditions holds.

(c) There is no path in H between u and v through $\{u, v\} \cup (S \cap V)$.
(d) There exists $l \in L$ with $D(l) = \{u, v\}$ and $R(l) \subseteq S \cap V$ which is not degraded during processing of $S \cap V$.

Let us show that both conditions imply that

(d′) there exists $k \in K$ with $D(k) = \{u, v\}$ and $R(k) \cap V \subseteq S$ which is not degraded during processing of S.

The implication (d) ⇒ (d′) follows easily from Observation 7.3 with help of Observation 5.2 (iii). To show that (d′) holds also in case of (c) we first verify that $\tau(u, v|B\|F) \neq \emptyset$. By Observation 7.1 where $(F, J) = (G, K)$, $\tilde{V} = V \cup B$, $\tilde{S} = S$ the fact that (u, v) is an edge in G^* implies that there exists a path ρ in G between u and v through $\{u, v\} \cup S$. The path ρ does not hit $S \cap V$ as otherwise by Observation 7.2, it can be shortened to a path in H through $\{u, v\} \cup (S \cap V)$ which contradicts (c). Thus, ρ is a path in G through $\{u, v\} \cup B$. If it is non-trivial, by Observation 5.2 (i) derive $\emptyset \neq \tau(u, v|B\|G) \subseteq \tau(u, v|B\|F)$. Otherwise (u, v) is an edge in G but not in H (by (c)) and by Step A1 of the annotation procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$ necessarily $\emptyset \neq \tau(u, v|B\|F)$. In either case, in Step A2 of the procedure an element $k \in K$ with $D(k) = \{u, v\}$ and $R(k) = \tau(u, v|B\|F)$ is newly created. By Observation 5.2 (iv) $R(k) \cap V = \emptyset$ and by Observation 7.4 k is not degraded during processing of S. Thus, the condition (d′) was verified. The element $k \in K$ from (d′) is therefore changed in the restriction step of the processing of S into an element $\tilde{k}$ with $D(\tilde{k}) = \{u, v\}$ and $R(\tilde{k}) = R(k) \setminus S$. Since (u, v) is an edge in G^* necessarily $R(k) \setminus S \neq \emptyset$, as otherwise (u, v) is removed from the graph in the reduction step of processing of S. Thus by Consequence 5.1 and Observation 7.6 write $\emptyset \neq R(k) \setminus S = \tau(u, v|(R(k) \cap V) \cup B\|G) \setminus S = \tau(u, v|((R(k) \cap V) \setminus S) \cup (B \setminus S)\|G^*)$. This completes the proof that $G_* = G^*$.

Suppose that $q \in K_*$. We must show that $q \in K^*$. By Consequence 5.1 applied to $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ derive that $R(q) = \tau(D(q)|(R(q) \cap V) \cup (B \setminus S)||G^*)$ (we know $R(q) \cap V = R(q) \cap (V \setminus S)$). We distinguish the following two cases.

(e) There exists $l \in L$ with $D(l) = D(q)$ and $R(l) \setminus S = R(q) \cap V$ which is not degraded during processing of $S \cap V$.

(f) Every $l \in L$ with $D(l) = D(q)$ and $R(l) \setminus S = R(q) \cap V$ is degraded during processing of $S \cap V$.

In the case (e) holds by Step A3 of the annotation procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$ there exists $k \in K$ with $D(k) = D(l)$ and $R(k) = \tau(D(l)|R(l) \cup B||F)$. By Observation 7.3 k is not degraded during processing of S. Thus, after the restriction step of processing of S an element k^* with $D(k^*) = D(k)$ and $R(k^*) = R(k) \setminus S$ is obtained. One can write according to Observations 5.5 and 7.6: $R(k^*) = R(k) \setminus S = \tau(D(l)|R(l) \cup B||F) \setminus S = \tau(D(l)|R(l) \cup B||G) \setminus S = \tau(D(l)|(R(l) \setminus S) \cup (B \setminus S)||G^*) = \tau(D(l)|(R(q) \cap V) \cup (B \setminus S)||G^*) = R(q) \neq \emptyset$. Thus, k^* is saved during the reduction step of processing of S and during subsequent removal of degraded elements. Therefore $k^* = q$ belongs to K^*.

In case the condition (f) holds, observe that $R(q) \cap V = \emptyset$. Indeed, otherwise by Observation 5.2 (iv) (iii) for $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ there exists $l^* \in L^*$ with $D(l^*) = D(q)$ and $R(l^*) = R(q) \cap V$ and this implies that there exists $l \in L$ with $D(l) = D(l^*)$ and $R(l^*) = R(l) \setminus S$ which is not degraded during processing of $S \cap V$. However, this contradicts the condition (f). The fact $R(q) \cap V = \emptyset$ then implies by Observation 7.6 and Observation 5.5 $R(q) = \tau(D(q)|B \setminus S||G^*) = \tau(D(q)|B||G) \setminus S = \tau(D(q)|B||F) \setminus S$. Moreover, by Observation 5.2 (iv) applied to $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ the element q was newly created. In particular, by Step A2 of the annotation procedure, the vertices in $D(q)$ are not an edge in H^*. Hence, Observation 7.1 where $(F, J) = (H, L)$, $\tilde{V} = V$, $\tilde{S} = S \cap V$ can be used to show that there is no path in H between nodes of $D(q)$ through $D(q) \cup (S \cap V)$. Indeed, otherwise by Observation 7.1 there exists $l \in L$ with $D(l) = D(q)$ and $R(l) \subseteq S \cap V$ which is not degraded during processing of $S \cap V$. Since $R(l) \setminus S = \emptyset = R(q) \cap V$ in this case, it contradicts the condition (f). Thus the vertices in $D(q)$ do not form an edge in H and $\emptyset \neq R(q) \subseteq \tau(D(q)|B||F)$ implies that in Step A2 of the annotation procedure $(G, K) = \underline{\text{Annot}}((H, L) : F)$ an element $k \in K$ with $D(k) = D(q)$ and $R(k) = \tau(D(q)|B||F)$ is newly created. By Observation 7.4, it is not degraded during processing of S. During the restriction step it is changed into an element k^* with $D(k^*) = D(k)$ and $R(k^*) = R(k) \setminus S = \tau(D(q)|B||F) \setminus S = R(q) \neq \emptyset$. Thus, k^* is saved during the reduction step and removal of degraded elements. Therefore $k^* = q$ belongs to K^*.

Suppose that $k^* \in K^*$. We must show that $k^* \in K_*$. By definition of (G^*, K^*) there exists $k \in K$ with $D(k) = D(k^*)$ and $R(k^*) = R(k) \setminus S$ such that k is not degraded during processing of S. Write by Consequence 5.1 and Observation 7.6 $R(k) \setminus S = \tau(D(k)|(R(k) \cap V) \cup B||G) \setminus S = \tau(D(k)|((R(k) \cap V) \setminus S) \cup (B \setminus S)||G^*) = \tau(D(k^*)|(R(k^*) \cap V) \cup (B \setminus S)||G^*)$. Now, we distinguish two cases.

In case $R(k^*) \cap V \neq \emptyset$ by Observation 5.2 (iv) there exists $l \in L$ with $D(l) = D(k)$ and $R(l) = R(k) \cap V$. According to Observation 7.3, l is not degraded during processing of $S \cap V$.

Thus, after the restriction step of processing of $S \cap V$ there exists an element l^* with $D(l^*) = D(l)$ and $R(l^*) = R(l) \setminus (S \cap V) = (R(k) \cap V) \setminus S = R(k^*) \cap V \neq \emptyset$. Therefore l^* is saved during the reduction step and subsequent removal of degraded elements. Since $l^* \in L^*$, in step A3 of the annotation procedure $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ an element $q \in K_*$ with $D(q) = D(l^*)$ and $R(q) = \tau(D(l^*)|R(l^*) \cup (B \setminus S)||G^*)$ is created. Substitute $R(l^*) = R(k^*) \cap V$ and use the formula above: $R(q) = \tau(D(k^*)|(R(k^*) \cap V) \cup (B \setminus S)||G^*) = R(k^*)$. Thus, $q = k^*$ and therefore $k^* \in K_*$.

In case $R(k^*) \cap V = \emptyset$ we show that the vertices in $D(k)$ do not form an edge in H^*. If $R(k) \cap V \neq \emptyset$, then by Observation 5.2 (iv), there exists $l \in L$ with $D(l) = D(k)$ and $R(l) = R(k) \cap V$. By Observation 7.3, l is not degraded during processing of $S \cap V$. Hence, after the restriction step an element l^* with $D(l^*) = D(l)$ and $R(l^*) = R(l) \setminus (S \cap V) = (R(k) \cap V) \setminus S = R(k^*) \cap V = \emptyset$. Thus, in the reduction step of processing of $S \cap V$ the edge between the vertices in $D(l^*) = D(l)$ is cancelled. Therefore the vertices do not form an edge in H^*. If $R(k) \cap V = \emptyset$, then we verify that the vertices $D(k)$ do not form an edge in H^* by contradiction. Indeed, otherwise by Observation 7.1 where $(F, J) = (H, L)$, $\tilde{V} = V$, $\tilde{S} = S \cap V$ derive that there exists a path in H between nodes of $D(k)$ through $D(k) \cap (S \cap V)$. In this subcase by Observation 5.2 (iv), k is newly created and one can apply Observation 7.4 for $(G, K) = \underline{\text{Annot}}((H, L) : F)$ to derive that k is degraded during processing of S. This contradicts the assumption about k. In either case, the vertices $D(k^*) = D(k)$ do not form an edge in H^* and $\emptyset \neq R(k^*) = \tau(D(k^*)|B \setminus S||G^*)$. Thus, in Step A2 of the annotation procedure $(G_*, K_*) = \underline{\text{Annot}}((H^*, L^*) : G^*)$ an element $q \in K_*$ with $D(q) = D(k^*)$ and $R(q) = \tau(D(k^*)|B \setminus S||G^*)$ is newly created. Hence $q = k^*$ and $k^* \in K_*$. □

7.2. Summary of Results about Annotation Algorithm

Throughout this subsection suppose that $G_1, \ldots, G_n$, $n \geq 1$ is a regular nest of undirected graphs and (G_i, K_i), $i = 1, \ldots, n$ the sequence of iterations of the corresponding annotation algorithm. Let us denote by V_i the set of nodes of G_i and put $B_i = V_i \setminus V_{i-1}$ for $i = 1, \ldots, n$ (by definition $V_0 = \emptyset$). The assumption of the regularity implies that the nest of the graphs is not changed during the annotation algorithm.

Consequence 7.1. Under the above assumptions above $gr\,(I(G_1) \cap \ldots \cup I(G_n)) \subseteq I(G_n, K_n)$.

Proof:
By Theorem 5.1, (G_i, K_i) is a regular annotated graph without degraded and void elements for $i = 1, \ldots, n$. Moreover, by Lemma 5.2 $I(G_{i-1}, K_{i-1}) \subseteq I(G_i, K_i)$ for $i = 2, \ldots, n$. However, it follows from the description of the membership algorithm and Observation 4.2 that $\forall i = 1, \ldots, n$ $I(G_i) \subseteq I(G_i, K_i)$. Altogether, $I(G_1) \cup \ldots \cup I(G_n) \subseteq I(G_n, K_n)$. Since $I(G_n, K_n)$ is a graphoid by Theorem 4.2, the desired conclusion follows easily. □

The converse inclusion will be proved by induction on the number of elements of K_n. We start with a simple observation.

Observation 7.7. Suppose that, under the assumption of this subsection, $K_n = \emptyset$. Then

$$I(G_n, K_n) \subseteq gr\,(I(G_1) \cup \ldots \cup I(G_n))\,.$$

Proof:
It follows from Step 3 of the membership algorithm that in case $K_n = \emptyset$ one has $I(G_n, K_n) = I(G_n)$. Thus $I(G_n) \subseteq I(G_1) \cup \ldots \cup I(G_n) \subseteq gr\,(I(G_1) \cup \ldots \cup I(G_n))$. □

The basis of the induction step is the following lemma.

Lemma 7.3. *Under the assumption of this subsection suppose that $p = (\{a,b\}, S)$ is a dominant element of K_n. Let us denote by (G_i^*, K_i^*) the annotated graph obtained from (G_i, K_i) by processing of $S \cap V_i$ and subsequent removal of degraded elements for $i = 1, \ldots, n$. Then (G_n^*, K_n^*) is a result of the annotation algorithm applied to $G_1^*, \ldots, G_n^*$ (in particular, it is a regular nest of undirected graphs), K_n^* has less elements than K_n, and $\forall j = 1, \ldots, n$ $I(G_j^*) \subseteq gr\,(I(G_1) \cup \ldots \cup I(G_j))$.*

Proof:
One can show by repeated application of Observation 5.2 (iv) (iii) that $\forall i = 1, \ldots, n$ either $S \cap V_i = R(k_i)$ for a dominant elements $k_i \in K_i$ or $S \cap V_i = \emptyset$. Thus, by Theorem 5.1, the assumptions of the commutativity lemma are fulfilled for every $i = 2, \ldots, n$ with $(H, L) = (G_{i-1}, K_{i-1})$, $F = G_i$, $V = V_{i-1}$, $B = B_i$ and $S \cap V_i$ instead of S. Repeated application of Lemma 7.2 makes it possible to show by induction on $j = 2, \ldots, n$ that (G_j^*, K_j^*) is a result of the annotation algorithm applied to $G_1^*, \ldots, G_j^*$.

Since (G_n^*, K_n^*) is obtained from (G_n, K_n) by processing of $S = R(p)$ the element p is surely removed in the reduction step of the processing. Hence, K_n^* has less number of elements than K_n (the number of elements cannot be increased by the considered change).

The last part is trivial for $j = 1$. Indeed, owing to Observation 5.6, $S \cap V_1 = \emptyset$ and since processing of the empty set makes no change $G_1^* = G_1$. For $1 < j \leq n$ denote by M_j the set of triplets over $V_j \setminus S$ which belong to $gr(I(G_1) \cup \ldots \cup I(G_j))$. Evidently, it is a graphoid over $V_j \setminus S$ and by Claim 2.1 applied to G_j^* it suffices to show that for every $c, d \in V_j \setminus S$, $c \neq d$ such that (c, d) is not an edge in G_j^* the triplet $t = (c, V_j \setminus Scd, d)$ belongs to M_j. We distinguish two cases. If there is no path in G_j between c and d through $\{c, d\} \cup (S \cap V_j)$, then $t \in I(G_j)$ and therefore $t \in M_j$.

If there exists a path in G_j between c and d through $\{c, d\} \cup (S \cap V_j)$, then it is unchanged during the degradation step of processing of $S \cap V_j$ and in the restriction step is shortened to the edge (c, d). Since (c, d) is not an edge in G_j^*, obtained after the reduction step, an void element having $\{c, d\}$ as domain was in the annotated graph before reduction. This implies that there exists an element $q \in K_j$ such that $D(q) = \{c, d\}$ and $R(q) \subseteq S \cap V_j$ which is not degraded during processing of $S \cap V_j$. Observe that $R(q) \neq \emptyset$ by Theorem 5.1 and $R(q) \cap V_1 = \emptyset$ by Observation 5.6. Set $m = \max\{i = 1, \ldots, j;\;\; R(q) \cap V_i = \emptyset\}$. We are going to apply Lemma 5.3 to the nest $G_1, \ldots, G_j$ $(n = j)$ and $S \cap V_j$ in place of S. It follows from the description of the annotation algorithm that there exists $1 \leq m \leq j - 1$ such that in the annotation procedure $(G_{m+1}, K_{m+1}) = \underline{\mathrm{Annot}}((G_m, K_m) : G_{m+1})$, an element $\tilde{q} \in K_{m+1}$ with

$D(\tilde{q}) = D(q)$ and $R(\tilde{q}) = R(q) \cap V_{m+1}$ is newly created. This implies (Step A2 of the annotation procedure) that (c, d) is not an edge in G_m. Suppose for a contradiction that there exists a path π in G_m between c and d through $\{c, d\} \cup (S \cap V_m)$. Then it is a non-trivial path and since G_m is a subgraph of G_j and $R(q) \cap V_m = \emptyset$ it is a path in G_j between c and d through $\{c, d\} \cup (S \cap V_j) \setminus R(q)$ which contradicts the fact that q is not degraded during processing of $S \cap V_j$. Hence $(c, V_m \backslash Scd, d) \in I(G_m)$. Moreover, it follows from the description of the annotation procedure together with Observation 5.1 (ii) that $\forall i = m+1, \ldots, j \;\; \tau(c, d|B_i||G_i) \subseteq R(q) \cap V_i$. Since $R(q) \subseteq S \cap V_j$ the condition (b) from Lemma 5.3 is also fulfilled. Thus, the lemma implies $t = (c, V_j \setminus Scd, d) \in gr\,(I(G_m) \cup \ldots \cup I(G_j))$. Hence, $t \in M_j$ and the conclusion was verified. □

Lemma 7.4. *Under the assumptions of this subsection*

$$I(G_n, K_n) \subseteq gr\,(I(G_1) \cup \ldots \cup I(G_n))\,.$$

Proof:
By Observation 7.7 the statement is valid if $K_n = \emptyset$. Suppose now that K_n has j elements, where $j \geq 1$, and we have already proved the statement of the lemma for every regular nest of undirected graphs $G_1^*, \ldots, G_n^*$ producing an annotated graph (G_n^*, K_n^*) where K_n^* has at most $j-1$ elements.

Take a triplet $t = (X, Z, Y) \in I(G_n, K_n)$; put $M = gr\,(I(G_1) \cup \ldots \cup I(G_n))$. If $XYZ \cap R(k) \neq \emptyset$ for every $k \in K_n$, then in the deletion step of the membership algorithm every element of K_n is deleted, and t is represented in G_n. Thus, $t \in I(G_n) \subseteq M$. If there exists $k \in K_n$ with $XYZ \cap R(k) = \emptyset$, then one can find (see Lemma 4.1) a dominant element $p \in K_n$ with $XYZ \cap R(p) = \emptyset$. One can construct a scenario ω for (G_n, K_n) suitable for testing (X, Z, Y) which starts with p. It follows from the description of the membership algorithm that t is represented in the graph (G_n^*, K_n^*) obtained from (G_n, K_n) by processing of $S = R(p)$.

Since (G_n^*, K_n^*) is a result of the annotation algorithm applied to $G_1^*, \ldots, G_n^*$ by Lemma 7.3, by the induction hypothesis $t \in I(G_n^*, K_n^*) \subseteq gr\,(I(G_1^*) \cup \ldots \cup I(G_n^*))$. However, Lemma 7.3 also says $I(G_j^*) \subseteq M$ for every $j = 1, \ldots, n$. Therefore $I(G_1^*) \cup \ldots \cup I(G_n^*) \subseteq M$ and since M is a graphoid $gr\,(I(G_1^*) \cup \ldots \cup I(G_n^*)) \subseteq M$. Hence, $t \in M$ and the induction step was made. □

Thus, Consequence 7.1 and Lemma 7.4 together imply that whenever $G_1, \ldots, G_n$ is a regular nest of undirected graphs and (G_i, K_i), $i = 1, \ldots, n$ are iterations of the corresponding annotation algorithm, then $I(G_n, K_n) = gr\,(I(G_1) \cup \ldots \cup I(G_n))$. So, Theorem 5.2 follows from this fact and from Lemma 5.4.

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