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STRUCTURAL SEMIGRAPHOIDS[1]

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The new approach to mathematical description of structures of stochastic conditional independence from [Studený, 1993a–c] is related to the classical approach by means of the notion of structural semigraphoid. It is shown how to realize the corresponding deductive mechanism to infer probabilistically valid consequences of input information about CI-structure, called facial implication. In the case of 4 attributes (random variables), structural semigraphoids are characterized in terms of inference rules.

INDEX TERMS: Conditional independence, dependency model, semigraphoid, structural face, skeletal imset, structural semigraphoid

INTRODUCTION

Although CI (=*conditional independence*) was studied in modern statistics many years ago [Dawid, 1979; Mouchart and Rolin, 1984], its importance for probabilistic expert systems was explicitly discerned and highlighted relatively lately [Pearl, 1986; Smith, 1989; Geiger and Pearl, 1989; Spiegelhalter and Lauritzen, 1990]. Pearl and Paz [1985] proposed the concept of dependency model to describe structures of CI for finite number of random variables. Unfortunately, their original hypothesis (see also [Pearl, 1986]) that models of CI-structures coincide with semigraphoids (a special class of dependency models introduced by Pearl and Paz [1985]) was refuted in [Studený, 1989a]. Later it was even found that models of CI-structures cannot be characterized as dependency models closed under finite number of inference rules [Studený, 1992]. This was lately strengthened by Geiger and Pearl [1993] who showed that even so-called disjunctive inference rules do not help.

These results inspired us to attempt to develop an alternative mathematical description of CI-structures for finite number of random variables, namely by means of so-called *structural faces*. This theory, presented as a series of 3 papers [Studený, 1993b–d], provides a deductive mechanism to infer probabilistically valid consequences of input information about CI-structure. This mechanism, called *facial implication* is much more powerful than the semigraphoid mechanism: it includes infinite number of inference rules! Nevertheless, it is finitely implementable from theoretical point of view. Of course, the new approach has its counterpart in the classical theory of dependency models: a special class of *structural semigraphoids* was defined as dependency models corresponding to structural faces.

In this paper we want to show (without mathematical technicalities and proofs contained in [Studený, 1993b–d]) how the practical realization of facial implication

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looks. To compare structural semigraphoids with general semigraphoids they are (in case of 4 attributes) characterized by means of five further inference rules.

NOTATION

Throughout this paper we deal with the following situation. A finite set N having at least two elements called the *basic set* is given. The class of *nontrivial* subsets of N i.e. subsets having at least two elements will be denoted by $\mathcal{U}$:

$$\mathcal{U} = \{S \subset N; \mathit{card}\, S \geq 2\}$$

The class of all subsets of N will be denoted by $\exp N$.
Review the symbols for number sets:

$\mathbf{Z}$	integers
$\mathbf{Z}^+ = \mathbf{Z} \cap \langle 0, \infty)$	nonnegative integers
$\mathbf{N}$	strictly positive integers (natural numbers)

§1 BASIC CONCEPTS

Let's begin with the concept of dependency model and semigraphoid. The definition from [Geiger and Pearl, 1989] is slightly modified here.

DEF 1 (dependency model, semigraphoid)
Denote by $T_*(N)$ the set of ordered triplets $\langle A, B, C\rangle$ where A, B, C are pairwise disjoint subsets of N and A, B nonempty. By *dependency model over N* we will understand every subset of $T_*(N)$.
A dependency model over N is called *semigraphoid* iff it is closed under following four inference rules:

$\langle A, B, C\rangle \to \langle B, A, C\rangle$ *symmetry*

$\langle A, B \cup C, D\rangle \to \langle A, C, D\rangle$ *decomposition*

$\langle A, B \cup C, D\rangle \to \langle A, B, C \cup D\rangle$ *weak union*

$[\langle A, C, D\rangle \;\&\; \langle A, B, C \cup D\rangle] \to \langle A, B \cup C, D\rangle$ *contraction.*

In case that $t \in T_*(N)$ is derivable from $I \subset T_*(N)$ by consecutive application of preceding inference rules we speak about *semigraphoid derivability.*

REMARK Pearl, Paz and Geiger call such inference rules axioms.

DEF 2 (model of CI-structure)
Consider a random vector $[\xi_i]_{i\in N}$ (where each random variable ξ_i takes its values in a nonempty finite set X_i).
Say that $[\xi_i]_{i\in N}$ *obeys a triplet* $\langle A, B, C\rangle \in T_*(N)$ iff $[\xi_i]_{i\in A}$ is conditionally independent of $[\xi_i]_{i\in B}$ given $[\xi_i]_{i\in C}$.
A dependency model over N is called the *model of CI-structure* of $[\xi_i]_{i\in N}$ iff it is exactly the set of triplets obeyed by $[\xi_i]_{i\in N}$.

REMARK Various another phrases were used in literature to say that I is the model of CI-structure (P denotes the distribution of the vector $[\xi_i]_{i \in N}$): 'I is induced by P' in [Ur and Paz, 1993], 'P is perfect for I' in [Geiger and Pearl, 1993], 'I is the conditional independence relation corresponding to P' in [Studený, 1992], 'I is probabilistically representable by P' in [Matúš, 1993].

It is well known that every model of CI-structure is a semigraphoid (see for example [Dawid, 1979]), but the converse does not hold [Studený, 1989b].

But semigraphoids have another important advantage. They give a simple deductive mechanism to infer probabilistically valid consequences of input information about CI-structure because the semigraphoid derivability implies probabilistic implication:

DEF 3 (probabilistic implication)
Suppose that $I \subset T_*(N)$ and $t \in T_*(N)$.
Say that I *probabilistically implies* t and write $I \models t$ iff every random vector that obeys I (i.e. all triplets from I) also obeys t.

REMARK Several phrases were used to name the previous situation, e.g. Geiger and Pearl said 't is logically implied by I' in [1989] or 't is entailed by I' in [1993].

§2 STRUCTURAL SEMIGRAPHOIDS

Description of CI-structures by means of finite number of inference rules appeared not to be perspective (because of infinite number of laws remains unused). Thus, the aim of the theory of faces was to afford a finitely implementable deductive mechanism which comprehends all (at that time) known properties of CI, especially infinite number of inference rules.

This design was realized. Indeed, structural faces give a class of dependency models which is much more closer to models of CI-structures than semigraphoids. These dependency models were named *structural semigraphoids* in [Studený, 1993c]. Unfortunately, it has lately appeared that even structural semigraphoids do not coincide with models of CI-structures. But the theory of faces allows modification, perhaps the latestly found laws of CI could be incorporated.

This section contains the definition of structural semigraphoid. But to reach it we need to recall several concepts from [Studený, 1993b–c]: imset, structural imset and (structural) face.

DEF 4 (imset, normalized imset)
Any function mapping the class of nontrivial sets $\mathcal{U}$ into $\mathbf{Z}$ will be called *imset* (*on* $\mathcal{U}$). An imset $u: \mathcal{U} \to \mathbf{Z}$ is *normalized* iff the collection of numbers $\{u(S); S \in \mathcal{U}\}$ has no common prime divisor.

REMARK The word imset is an abbreviation for **i**nteger-valued **multiset**. The term 'multiset' is widely used in combinatorial theory—see [Aigner, 1979].

The class of all imsets is too wide to describe CI-structures. Therefore a special class of structural imsets was introduced for this purpose. Their rather technical definition is below. Two equivalent characterizations (maybe more lucid for some readers) are mentioned in §3 and §5. Note that the adjective 'structural' was chosen because these imsets should correspond to CI-structures.

DEF 5 (semielementary imset, structural imset)
Having a couple of sets $U, V \subset N$ with $U \setminus V \neq \emptyset \neq V \setminus U$ consider the following function $\bar{u} : exp\, N \to \mathbf{Z}$:

$\bar{u}(U \cup V) = \bar{u}(U \cap V) = +1 \quad \bar{u}(U) = \bar{u}(V) = -1$

$\bar{u}(W) = 0$ for remaining $W \subset N$.

The restriction of $\bar{u}$ to $\mathcal{U}$ is called a *semielementary imset* corresponding to the couple $\{U, V\}$.
An imset u is called *structural* iff there exists a collection of semielementary imsets (necessarily finite) $\{v_\alpha;\ \alpha \in I\}$, coefficients $k_\alpha \in \mathbf{Z}^+$ and $n \in \mathbf{N}$ such that $n \cdot u = \sum_{\alpha \in I} k_\alpha \cdot v_\alpha$.

As written above, just the concept of face was introduced in [Studený, 1993c] to describe CI-structures. The definition is recalled here:

DEF 6 (structural face)
A set of structural imsets F is called a *(structural) face* iff it satisfies the following conditions:

(F.0) $0 \in F$ *nontriviality*

(F.1) $u, v \in F \Rightarrow u + v \in F$ *composition*

(F.2) u, v structural imsets $u + v \in F \Rightarrow u, v \in F$ *decomposition.*

REMARK Note that the terminology (the name face) found its motivation in the theory of convex polytopes, for details see [Studený, 1993c].

DEF 7 (structural semigraphoid)
First, introduce a primary mapping i which identifies elements of $T_*(N)$ with semielementary imsets: to each triplet $\langle A, B, C\rangle$ assign the semielementary imset corresponding to the couple $\{A \cup C, B \cup C\}$.
Then, having a structural face F the corresponding dependency model I is defined as follows:

$$\langle A, B, C\rangle \in I \quad \text{iff} \quad i(\langle A, B, C\rangle) \in F.$$

Any such dependency model will be called a *structural semigraphoid.*

As indicated by terminology, every dependency model corresponding to a face is indeed a semigraphoid. Moreover, it holds:

THEOREM 1 Every model of CI-structure is a structural semigraphoid.

This result is proved in [Studený, 1993c] as Consequence 2.9. However, there exist semigraphoids which are not structural (see Example 2.1 in the same paper). Moreover, as explained there in Remark 2.3 structural semigraphoids cannot be characterized as dependency models closed under finite number of inference rules. Therefore difference between (general) semigraphoids and structural semigraphoids is indeed qualitative.

§3 SKELETON

The following task is crucial for computer implementation of our theory: to be able to recognize structural imsets in finitely many steps.

The presented definition (Def 5) is not suitable for this purpose as there is no limitation for the number $n \in \mathbf{N}$. Nevertheless, the following result enables us to solve this problem from theoretical point of view.

THEOREM 2 There exists the least *finite* set S of normalized imsets (which are even nonegative) such that for every imset u it holds:

$$[u \text{ is structural}] \Leftrightarrow [\forall s \in S \ \langle s, u\rangle \geq 0]$$

where $\langle s, u\rangle = \Sigma_{S \in \mathcal{U}} s(S) \cdot u(S)$.

This result is proved in [Studený, 1993c] as Consequence 2.5.

DEF 8 (skeleton)
The set S from the preceding theorem is called the *(structural) skeleton*, their elements *skeletal imsets*.

If a computer has all skeletal imsets in its memory, it can easily judge whether a given imset u is structural: simply to check the nonnegativity of numbers $\langle s, u\rangle$ for all skeletal imsets s.

Thus, our task is solved from theoretical point of view. But, from practical point of view, we still need to find the skeleton. So far, I have no sufficiently convenient characterization of skeletal imsets giving an algorithm finding the skeleton for every cardinality of the basic set. Note (without proof) that this problem is equivalent to the problem of finding extremal convex games from game theory (for this concept see [Shapley, 1972]).

The number of skeletal imsets increases with the cardinality of the basic set. In case *card* $N = 3$ the skeleton has 5 imsets. The following table gives their list for $N = \{1, 2, 3\}$, the proof is in [Studený, 1993d] in Example 3.1.

	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}
s_1	0	0	0	1
s_2	1	0	0	1
s_3	0	1	0	1
s_4	0	0	1	1
s_5	1	1	1	2

In case *card* $N = 4$ the skeleton has 37 elements, their list for $N = \{1, 2, 3, 4\}$ is in the table below.

	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	N
s_1	0	0	0	0	0	0	0	0	0	0	1
s_2	0	0	0	0	0	0	1	0	0	0	1
s_3	0	0	0	0	0	0	0	1	0	0	1
s_4	0	0	0	0	0	0	0	0	1	0	1
s_5	0	0	0	0	0	0	0	0	0	1	1
s_6	0	0	0	0	0	0	0	1	1	1	2
s_7	0	0	0	0	0	0	1	0	1	1	2
s_8	0	0	0	0	0	0	1	1	0	1	2
s_9	0	0	0	0	0	0	1	1	1	0	2

	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	N
s_{10}	1	0	0	0	0	0	1	1	0	0	1
s_{11}	0	1	0	0	0	0	1	0	1	0	1
s_{12}	0	0	1	0	0	0	0	1	1	0	1
s_{13}	0	0	0	1	0	0	1	0	0	1	1
s_{14}	0	0	0	0	1	0	0	1	0	1	1
s_{15}	0	0	0	0	0	1	0	0	1	1	1
s_{16}	0	0	0	0	0	0	1	1	1	1	2
s_{17}	1	0	0	0	0	0	1	1	1	1	2
s_{18}	0	1	0	0	0	0	1	1	1	1	2
s_{19}	0	0	1	0	0	0	1	1	1	1	2
s_{20}	0	0	0	1	0	0	1	1	1	1	2
s_{21}	0	0	0	0	1	0	1	1	1	1	2
s_{22}	0	0	0	0	0	1	1	1	1	1	2
s_{23}	0	0	1	0	1	1	1	2	2	2	3
s_{24}	0	1	0	1	0	1	2	1	2	2	3
s_{25}	1	0	0	1	1	0	2	2	1	2	3
s_{26}	1	1	1	0	0	0	2	2	2	1	3
s_{27}	1	1	0	1	0	0	2	1	1	1	2
s_{28}	1	0	1	0	1	0	1	2	1	1	2
s_{29}	0	1	1	0	0	1	1	1	2	1	2
s_{30}	0	0	0	1	1	1	1	1	1	2	2
s_{31}	0	1	1	1	1	1	2	2	2	2	4
s_{32}	1	0	1	1	1	1	2	2	2	2	4
s_{33}	1	1	0	1	1	1	2	2	2	2	4
s_{34}	1	1	1	0	1	1	2	2	2	2	4
s_{35}	1	1	1	1	0	1	2	2	2	2	4
s_{36}	1	1	1	1	1	0	2	2	2	2	4
s_{37}	1	1	1	1	1	1	2	2	2	2	3

The completeness of that list is proved in [Studený, 1991], but as I recognized lately in [Shapley, 1972] these functions were already determined by S. A. Cook in 1965.

The following easy example illustrates our method how to recognize nonstructural imsets.

EXAMPLE 1 Consider $N = \{1, 2, 3\}$ and imsets u_1, u_2:

$$u_1(\{1, 2\}) = u_1(\{1, 3\}) = u_1(\{2, 3\}) = +1 \qquad u_1(N) = 0$$

$$u_2(\{1, 2\}) = u_2(\{1, 3\}) = u_2(\{2, 3\}) = -1 \qquad u_2(N) = +1$$

and the task to judge whether they are structural. To this end make the corresponding table of numbers $\langle s, u\rangle$ for skeletal imsets.

	s_1	s_2	s_3	s_4	s_5
u_1	0	1	1	1	3
u_2	1	0	0	0	−1

Thus, according to Theorem 2 the imset u_1 is structural while the imset u_2 is not. Indeed, u_1 can be written as the sum of semielementary imsets listed in the following table (the last column contains the corresponding couple of sets—see Def 5).

	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}	couple
v_1	1	0	0	0	{1}, {2}
v_2	0	1	0	0	{1}, {3}
v_3	0	0	1	0	{2}, {3}

§4 FACIAL IMPLICATION

As mentioned above structural semigraphoids have no finite 'simple' axiomatic characterization. Nevertheless, they can be endowed by another finitely implementable deductive mechanism, namely they can utilize the facial implication developed for structural imsets.

DEF 9 (facial implication)
Having a set of structural imsets L and another structural imset v say that L *facially implies* v and write $L \mapsto v$ iff there exists a finite subset $L' \subset L$ and coefficients $k_w \in \mathbf{Z}^+$ ($w \in L'$) such that $(\Sigma_{w \in L'} k_w \cdot w - v)$ is a structural imset.

To explain the terminology note that the implication $L \mapsto v$ can be equivalently defined as follows: whenever a (structural) face contains L then v belongs to that face too (see Lemma 2.2 in [Studený, 1993c]). The following result is proved in the same paper as Consequence 2.8.

THEOREM 3 Whenever $\mathrm{I} \subset \mathrm{T}_*(\mathrm{N})$ and $\mathrm{t} \in \mathrm{T}_*(\mathrm{N})$ then

$$i(I) \mapsto i(t) \quad \text{entails} \quad I \models t$$

(i denotes the primary mapping from Def 7).

Thus, the facial implication gives a tool to derive probabilistically valid consequences of input information about CI-structure. The following example illustrates how facial implication can be verified using the original definition.

EXAMPLE 2 Consider $N = \{1, 2, 3, 4\}$ and imsets:

$$u_1(\{1,2,3\}) = +1 \qquad u_1(\{1,3\}) = u_1(\{2,3\}) = -1 \qquad u_1(Z) = 0 \quad \text{otherwise}$$
$$u_2(\{1,3,4\}) = +1 \qquad u_2(\{1,4\}) = u_2(\{3,4\}) = -1 \qquad u_2(Z) = 0 \quad \text{otherwise}$$
$$u_3(\{1,2,4\}) = +1 \qquad u_3(\{1,2\}) = u_3(\{2,4\}) = -1 \qquad u_3(Z) = 0 \quad \text{otherwise}$$

and ask whether the imset

$$u_4(\{1,2,3\}) = +1 \qquad u_4(\{1,2\}) = u_4(\{2,3\}) = -1 \qquad u_4(Z) = 0 \quad \text{otherwise}$$

is facially implied by $\{u_1, u_2, u_3\}$. The following table

	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	N
u_1	0	−1	0	−1	0	0	1	0	0	0	0
u_2	0	0	−1	0	0	−1	0	0	1	0	0
u_3	−1	0	0	0	−1	0	0	1	0	0	0
$\Sigma_{i=1}^3 u_i$	−1	−1	−1	−1	−1	−1	1	1	1	0	0
u_4	−1	0	0	−1	0	0	1	0	0	0	0
u_5	0	0	−1	0	−1	0	0	1	0	0	0
u_6	0	−1	0	0	0	−1	0	0	1	0	0

shows that $(u_1 + u_2 + u_3) = u_4 + (u_5 + u_6)$ where u_5, u_6 are evidently semielementary imsets. Thus, according to Def 9 get $\{u_1, u_2, u_3\} \mapsto u_4$.

Nevertheless, the method in preceding example is too clumsy. At first we have to guess coefficients k_w and then we have to be able to "decompose" $(\Sigma k_w \cdot w - v)$ into semielementary imsets. Moreover, it gives no instruction how to refute facial implication, it can give only positive answer. But in case that the skeleton is at our disposal it is no problem to use the following easy criterion (it can be derived as a consequence of Theorem 2, see also [Studený, 1993c], Assertion 2.1).

CONSEQUENCE 1 Having a set of structural imsets L and another structural imset v the implication $L \mapsto v$ takes place iff

$$\forall s \in S \ \langle s, v\rangle > 0 \Rightarrow [\langle s, v\rangle > 0 \quad \text{for some} \quad u \in L]$$

(S denotes the skeleton).

This method can be also used to prove that the facial implication does not take place. The following example illustrates it.

EXAMPLE 3 Consider $N = \{1, 2, 3, 4\}$, the following imsets

$$u_1(N) = u_1(\{3, 4\}) = +1, u_1(\{1, 3, 4\}) = u_1(\{2, 3, 4\}) = -1, u_1(Z) = 0 \quad \text{otherwise}$$

$$u_2(N) = u_1(\{1, 2\}) = +1, u_2(\{1, 2, 3\}) = u_1(\{1, 2, 4\}) = -1, u_2(Z) = 0 \quad \text{otherwise}$$

$$u_3(\{1, 3\}) = +1, \qquad u_3(Z) = 0 \quad \text{otherwise}$$

$$u_4(\{2, 4\}) = +1, \qquad u_4(Z) = 0 \quad \text{otherwise}$$

and ask whether the fifth imset

$$u_5(\{1, 2\}) = +1, \qquad u_5(Z) = 0 \quad \text{otherwise}$$

is facially implied by $\{u_1, u_2, u_3, u_4\}$. It is no problem to see that $\langle s, u_5\rangle > 0$ only for skeletal imsets listed in the following table of numbers $\langle s, u\rangle$:

$\langle s, u\rangle$	u_5	u_1	u_2	u_3	u_4
s_{10}	1	1	0	0	0
s_{17}	1	0	1	0	0
s_{25}	1	0	0	0	1
s_{26}	1	0	0	1	0
s_{27}	1	0	0	1	0
s_{28}	1	0	0	0	1
s_{32}	1	1	1	0	1
s_{33}	1	1	1	1	1
s_{34}	1	1	1	1	1
s_{35}	1	1	1	1	0
s_{36}	1	0	1	1	1
s_{37}	1	0	0	1	1

Evidently, the condition from Consequence 1 is satisfied and therefore $\{u_1, u_2, u_3, u_4\} \mapsto u_5$. On the other hand, $\{u_1, u_2, u_3\} \not\mapsto u_5$ since $\langle s_{25}, u_5\rangle > 0$ but $\langle s_{25}, u_i\rangle = 0$ for $i = 1, 2, 3$.

The method described in preceding examples can be used to derive further inference rules valid for structural semigraphoids and therefore for models of CI-structures. Namely, having a hypothesis that a set $I \subset T_*(N)$ implies $t \in T_*(N)$ firstly translate elements of I and t to semielementary imsets (by means of primary mapping from Def 7) and then try to verify $i(I) \mapsto i(t)$. By Theorem 3 it gives $I \models t$. This procedure can be used to derive the following 5 inference rules holding for structural semigraphoids.

(A.3) $[\langle A, B, C \cup D\rangle \;\&\; \langle C, D, A\rangle \;\&\; \langle C, D, B\rangle \;\&\; \langle A, B, \emptyset\rangle] \leftrightarrow$
$\leftrightarrow [\langle C, D, A \cup B\rangle \;\&\; \langle A, B, C\rangle \;\&\; \langle A, B, D\rangle \;\&\; \langle C, D, \emptyset\rangle]$

(A.4) $[\langle A, B, C \cup D\rangle \;\&\; \langle A, D, B\rangle \;\&\; \langle C, D, A\rangle \;\&\; \langle B, C, \emptyset\rangle] \leftrightarrow$
$\leftrightarrow [\langle A, D, B \cup C\rangle \;\&\; \langle A, B, D\rangle \;\&\; \langle B, C, A\rangle \;\&\; \langle D, C, \emptyset\rangle]$

(A.5) $[\langle A, C, D\rangle \;\&\; \langle B, D, C\rangle \;\&\; \langle B, C, A\rangle \;\&\; \langle A, D, B\rangle] \leftrightarrow$
$\leftrightarrow [\langle A, D, C\rangle \;\&\; \langle B, C, D\rangle \;\&\; \langle B, D, A\rangle \;\&\; \langle A, C, B\rangle]$

(A.6) $[\langle A, B, C\rangle \;\&\; \langle A, C, D\rangle \;\&\; \langle A, D, B\rangle] \leftrightarrow$
$\leftrightarrow [\langle A, C, B\rangle \;\&\; \langle A, D, C\rangle \;\&\; \langle A, B, D\rangle]$

(A.7) $[\langle A, B, C \cup D\rangle \;\&\; \langle C, D, A \cup B\rangle \;\&\; \langle A, C\ \emptyset\rangle \;\&\; \langle B, D, \emptyset\rangle] \leftrightarrow$
$\leftrightarrow [\langle A, C, B \cup D\rangle \;\&\; \langle B, D, A \cup C\rangle \;\&\; \langle A, B, \emptyset\rangle \;\&\; \langle C, D, \emptyset\rangle]$

For instance, in Example 2 we have derived the implication
$[\langle 1, 2, 3\rangle \;\&\; \langle 1, 3, 4\rangle \;\&\; \langle 1, 4, 2\rangle] \rightarrow \langle 1, 3, 2\rangle$ i.e. part of (A.6), in Example 3
$[\langle 1, 2, 34\rangle \;\&\; \langle 3, 4, 12\rangle \;\&\; \langle 1, 3, \emptyset\rangle \;\&\; \langle 2, 4, \emptyset\rangle] \rightarrow \langle 1, 2, \emptyset\rangle$ i.e. part of (A.7).
Note that inference rules listed above characterize structural semigraphoids in case *card* $N = 4$. The verification was made by means of a computer. Mention that the number of all semigraphoids in that case is 26 424 while the number of structural semigraphoids is 22 108. By the way, in case *card* $N = 3$ we have 22 semigraphoids and every semigraphoid is structural.

§5 REMARKS

As mentioned in §2 there exist structural semigraphoids which are not models of CI-structures. In particular, facial implication is not equivalent to probabilistic implication. I must emphasize that the essential step to this discovery was made by my colleague František Matúš [1993] who found that (said in his terminology) there exists a matroid which is not probabilistically representable. Knowing this I succeeded to find further independent properties of models of CI-structures which do not hold for structural semigraphoids:

(B.1) $[\langle A, B, C\rangle \;\&\; \langle A, B, D\rangle \;\&\; \langle B, C, A\rangle \;\&\; \langle C, D, \emptyset\rangle] \rightarrow \langle B, A \cup C, \emptyset\rangle$

(B.2) $[\langle A, B, D\rangle \;\&\; \langle A, C, B\rangle \;\&\; \langle B, C, A\rangle \;\&\; \langle C, D, \emptyset\rangle] \rightarrow \langle A \cup B, C, \emptyset\rangle$

The proofs of these results are in manuscript still, we (i.e. I and Matúš) plan to publish it later (hopefully after having examined all models of CI-structures in the case *card* $N = 4$).

Let us point out to some assets of the theory of faces. This approach makes it possible to prove further laws for CI, i.e. to verify probabilistic soundness of conjectured inference rules (this simple method was outlined in Example 2, more explicitly see Example 3.7 in [Studený, 1993d]). Moreover, it allows input of information about CI-structure in various forms (individual CI-statements, Markov nets, influence diagrams). Information from different sources can be easily combined without loss of information about CI-structure (for details see [Studený, 1993d] Examples 3.3–3.6).

According to the result from [Studený, 1993a] every facial model of CI-structure can be equivalently described by means of the validity of certain product formula. This can be understood as the first step to a justified global interpretation of these models of CI-structures. In fact, some links to well-known hierarchical loglinear models are made.

In [Studený, 1993d] two possible representations of faces were derived:

Representation by means of generating imsets For every (structural) face F there exists a structural imset u such that F is the least face containing u i.e. the face F is generated by u (see Theorem 2.2 there).

Representation in terms of the skeleton Every (structural) face has the form: $F = \{u$ structural imset; $\forall s \in T\ \langle s, u\rangle = 0\}$ where $T \subset S$ is a subset of the skeleton (this is treated as Theorem 2.3 there).

Let us mention further equivalent definition of structural imset, namely by means of the class of so-called completely convex set functions (see [Studený, 1993d], Theorem 2.4):

$[u$ is structural$]$ $\Leftrightarrow$ $[\langle m, u\rangle \geq 0$ for every completely convex set function $m]$.

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