Linear criterion for testing the extremity of an exact game based on its finest min-representation

Milan Studený*, Václav Kratochvíl

Czech Academy of Sciences, the Institute of Information Theory and Automation
Prague, Pod Vodárenskou věží 4, 18208, Czech Republic

Abstract

A game-theoretical concept of an exact (cooperative) game corresponds to the notion of a discrete coherent lower probability, used in the context of imprecise probabilities. The collection of (suitably standardized) exact games forms a pointed polyhedral cone and the paper is devoted to the recognition of extreme rays of that cone, whose generators are called *extreme exact games*. We give a necessary and sufficient condition for an exact game to be extreme. Our criterion leads to solving a simple linear equation system determined by a certain min-representation of the game. It has been implemented on a computer and a web-based platform for testing the extremity of an exact game is available, which works with a modest number of variables.

The paper also deals with different *min-representations* of a fixed exact game μ, which can be compared with the help of the concept of a *tightness structure* (of a min-representation) introduced in the paper. The collection of tightness structures (of min-representations of μ) is shown to be a finite lattice with respect to a refinement relation. We give a method to obtain a min-representation with the finest tightness structure, which construction comes from the coarsest standard min-representation of μ given by the (complete) list of vertices of the core (polytope) of μ. The newly introduced criterion for exact extremity is based on the finest tightness structure.

Keywords: extreme exact game, coherent lower probability, core, supermodular game, finest min-representation, oxytrophic game

1. Motivation and overview of former results

The notion of a *coherent lower probability* and that of an induced *credal set* (of discrete probability distributions) are traditional topics of interest in the theory of imprecise probabilities. These notions correspond to game-theoretical concepts of an *exact game* and its *core* (polytope), widely used in the context of cooperative coalition games. The analogy is even broader: a lower probability avoiding sure loss corresponds to a weaker concept of a balanced game while a *2-monotone lower probability* (= capacity) corresponds to a stronger concept of a *supermodular game*, also named a convex game in game-theoretic community.

The discrete case is considered here: the sample space (= frame of discernment) for distributions is a fixed finite set N having at least two elements. The elements of N correspond to players in the context of cooperative game theory and to random variables in yet another context of probabilistic conditional independence structures. The collection of coherent lower probabilities on N, where $n = |N|$, is a polytope in a 2^n-dimensional real vector space, while the set of non-negative exact games is a pointed polyhedral cone whose extreme rays are generated just by extreme points of that polytope. In fact, the set of coherent lower probabilities can be viewed as the intersection of the cone of non-negative exact games μ with a normalizing hyperplane specified by $\mu(N) = 1$.

This paper offers a method to recognize whether a ray is extreme in the cone of exact games, which implicitly gives a method to recognize extreme coherent lower probabilities. From the geometric point of view, the problem of

*Corresponding author
Email addresses: studeny@utia.cas.cz (Milan Studený), velorex@utia.cas.cz (Václav Kratochvíl)

recognition of extreme rays is interesting in itself. Thus, it is natural that it has been treated earlier, both in the context of game theory and in the context of imprecise probabilities.

1.1. Extremity criteria in game theory

Some effort to develop criteria to recognize the extremity of an exact game was exerted earlier by Rosenmüller [16, § 4 of chapter 5] in his book on game theory. He offered one necessary and one sufficient condition for the extremity based on a *min-representation* of the exact game; however, these conditions have a limited scope of use because they are applicable only in quite special situations. In this paper we follow the idea of a min-representation and propose a more general criterion whose input is the list of vertices of the *core*, which serves as a *standard min-representation* of any exact game. Our condition is necessary and sufficient for the extremity of an exact game. Specifically, our extremity criterion can use any of the so-called *finest min-representations* of an exact game; such a min-representation can easily be obtained from the standard one. Note that, for certain exact games, the standard min-representation is the only (finest) min-representation. These games were named *oxytrophic* in [16].

1.2. Extremity in context of imprecise probabilities

Analogous problems were studied in connection with imprecise probabilities. Questions raised by Maass [11] motivated Quaeghebeur and de Cooman [14] to become interested in *extreme (coherent) lower probabilities* and to compute these in the case of small $n = |N|$. Antonucci and Cuzzolin [1] considered an enlarging transformation of a credal set with a finite number of extreme points, when the respective (coherent) lower probability is computed and then a larger credal set is induced by the lower probability. Their second step, namely representing a coherent lower probability by the vertices of the induced credal set, corresponds to our standard min-representation of an exact game.

The theme of characterizing the extreme lower probabilities from [14] motivated an even more general problem of characterizing the *extreme lower previsions* discussed by De Bock and de Cooman [2]; they related extreme lower previsions to indecomposable compact convex sets in a finite-dimensional space.

It is always useful to be aware of the correspondence between concepts from different areas. For instance, Wallner [21] confirmed a conjecture raised by Weichselberger that the credal set induced by a (coherent) lower probability has at most $n!$ vertices. Nonetheless, the same result was achieved already by Derks and Kuipers [6] in the context of cooperative game theory. They also made an interesting observation that whenever a core of an exact game has $n!$ vertices then it has the maximal number of $2^n - 2$ facets and gave an example of a game in the relative interior of the exact cone whose respective core does not have the maximal number of $n!$ vertices.

Note in this context that the polytopes which are cores of exact games are even more general than the so-called generalized permutohedra introduced in [13], which are also known to have at most $n!$ vertices; see [19, Remark 12].

1.3. The case of the supermodular cone

The criterion we offer here is a modification of the criterion from [19], where a necessary and sufficient condition was provided for a supermodular game to be extreme in the cone of (suitably standardized) supermodular games. That result was motivated by the research on conditional independence structures [17], in which context extreme supermodular games encode submaximal structural conditional independence models. The supermodular criterion leads to solving a simple linear equation system determined by certain combinatorial structure (of the core), which concept was pinpointed earlier by Kuipers et. al. [9]. An analogous combinatorial concept has also appeared in the context of imprecise probabilities: Bronevich and Rozenberg [3, Proposition 5] in their description of non-extreme 2-monotone lower probabilities use "the collection of maximal lattices on which the 2-monotone measure is additive", which collection seems to coincide with the above-mentioned concept of *core structure* from [9, 19]. More specifically, on basis of our consultation of [3], we think that such coincidence holds but a complete proof of this conjecture of ours would require some work because those two concepts are defined in different terms.

1.4. The case of the exact cone

The criterion for the extremity in the exact cone from the present paper can be viewed as a generalization of the former supermodular criterion [19]. One can assign an analogous linear equation system to any sensible min-representation of an exact game. The equation system is determined by the *tightness structure* of the min-representation, which is a combinatorial concept directly generalizing that of a core structure from [9, 19]. Nevertheless, in the case of a general min-representation some modification is needed. More specifically, the equation system assigned to a general min-representation can have some non-zero dummy solutions, which never occur in case of the standard min-representation. Fortunately, the dimension of the space of these dummy solutions can be computed easily. Thus, in the general case, one has to correct the dimension of the space of solutions by subtracting the dimension of the space of dummy solutions to get the so-called *essential dimension*, which is the right indicator of the extremity.

We have implemented both criteria and provide web-based platforms for testing the extremity of a supermodular/exact game in the respective cone for a modest number of variables. Of course, this can also be used to test the extremity of coherent lower probabilities on small sample spaces. We have, however, intentionally chosen to work with integer-valued games because this approach allows one to utilize the profits of integer arithmetic implementation.

1.5. The structure of the paper

The present paper extends a former conference paper [18]; however, it brings additional results, in particular, a necessary and sufficient condition for the exact extremity. Moreover, the terminology from [18] has slightly been modified in the present paper to be more informative (this concerns later Definitions 3 and 7).

The structure of the paper is as follows. In the next section (§ 2) we recall basic concepts and facts. In § 3 the concept of a min-representation of an exact game is introduced. Different min-representations of the same (exact) game can be compared using a relation of *refinement* introduced in § 4; the concept of a *finest min-representation* is introduced there. After that, our criterion is introduced, explained and illustrated by an example in § 5. The question of possible uniqueness of the min-representation of a game is discussed in § 6. In Conclusions (§ 7) we give a few remarks based on our computational experiments and mention open tasks. The Appendix contains the proofs.

2. Notation, basic definitions and facts

Throughout the paper we assume that the reader is familiar with basic concepts in polyhedral geometry, like a polytope (= bounded polyhedron), polyhedral cones, their dimension, faces, facets, and vertices.

Let N be a finite non-empty set of *variables*, $|N| \geq 2$; its power set will be denoted by $\mathcal{P}(N) := \{S \,:\, S \subseteq N\}$. The symbol $\mathbb{R}^N$ will denote the set of real vectors whose components are indexed by elements of N. Analogously, $\mathbb{R}^{\mathcal{P}(N)}$ is the set of real functions on $\mathcal{P}(N)$ (= vectors with components indexed by subsets of N). Given a set $S \subseteq N$, the symbol χ_S will denote its zero-one indicator vector in $\mathbb{R}^N$: $\chi_S(i) = 1$ for $i \in S$ and $\chi_S(j) = 0$ for $j \in N \setminus S$. Given $v, x \in \mathbb{R}^N$, their scalar product will be $\langle v, x\rangle := \sum_{i\in N} v_i \cdot x_i$. The symbol $\dim(L)$ will denote the dimension of a linear space $L \subseteq \mathbb{R}^N$, the symbol $\mathrm{Lin}\,(*)$ the linear hull (in the respective vector space). The Minkowski sum of sets $X, Y \subseteq \mathbb{R}^N$ will be denoted by $X \oplus Y := \{x + y \,:\, x \in X \;\&\; y \in Y\}$.

2.1. Game-theoretical concepts

By a *game* is meant a set function $\mu \in \mathbb{R}^{\mathcal{P}(N)}$ with $\mu(\emptyset) = 0$. Note that this is a shortened version of a longer term *transferable utility cooperative game* one can meet in game theory [16]. Let's call a game μ *normalized* if $\mu(N) = 1$.

Definition 1 (core, exact game, supermodular game)**.**
Let $\mu : \mathcal{P}(N) \to \mathbb{R}$ be a game. Its *core* is a polytope in $\mathbb{R}^N$ defined by

$$C(\mu) \;:=\; \{\, x \in \mathbb{R}^N \,:\, \sum_{i\in N} x_i = \mu(N) \;\;\&\;\; \forall\, S \subseteq N \quad \sum_{i\in S} x_i \geq \mu(S)\,\}\,.$$

The symbol $\mathsf{ext}\, C(\mu)$ will be used to denote the set of extreme points of $C(\mu)$; note that an alternative term for an extreme point is a *vertex*. A game μ is *balanced* if $C(\mu) \neq \emptyset$. A balanced game is called *exact* if

$$\forall\, S \subseteq N \;\; \exists\, x \in C(\mu) \qquad \sum_{i\in S} x_i = \mu(S)\,.$$

A game μ is *supermodular* if it satisfies the supermodularity inequalities

$$\forall\, C, D \subseteq N \qquad \mu(C) + \mu(D) \;\leq\; \mu(C \cup D) + \mu(C \cap D)\,.$$

A game μ will be called *ℓ-standardized* (where the letter ℓ stands for *lower* standardization) if $\mu(S) = 0$ for any $S \subseteq N$ with $|S| \leq 1$. Let us denote the class of exact ℓ-standardized games by $\mathsf{E}_\ell(N)$.

Note that ℓ-standardized games are named zero-normalized in game theory, but we feel that "standardization" is more suitable word here since "normalization" is more related to multiplication by a constant. A well-known fact is that any supermodular game, named traditionally *convex* in game theory, is exact [5, § 4]. The fact that, for any set $S \subseteq N$, $\{ x \in C(\mu) \,:\, \sum_{i\in S} x_i = \mu(S) \}$ is a face of $C(\mu)$ allows one to observe that any exact game μ satisfies a formally stronger condition

$$\forall\, S \subseteq N \;\; \exists\, x \in \operatorname{ext} C(\mu) \qquad \sum_{i\in S} x_i = \mu(S)\,. \tag{1}$$

Indeed, every face of a polytope is the convex hull of extreme points of the whole polytope contained in the face. A necessary condition for the exactness of a game μ is that it is *superadditive*:

$$\forall\, A, B \subseteq N \;\; A \cap B = \emptyset \quad \mu(A) + \mu(B) \leq \mu(A \cup B)\,.$$

Indeed, given disjoint $A, B \subseteq N$ there exists $x \in C(\mu)$ with $\mu(A\cup B) = \sum_{i\in A} x_i + \sum_{i\in B} x_i$ and one has both $\sum_{i\in A} x_i \geq \mu(A)$ and $\sum_{i\in B} x_i \geq \mu(B)$. In particular, any ℓ-standardized exact game is *non-decreasing*, which means $\mu(S) \leq \mu(T)$ whenever $S \subseteq T \subseteq N$, and, hence, *non-negative*.

It can be derived from results in [10] that the collection of exact games is a rational polyhedral cone, which means it can be specified by finitely many inequalities with rational coefficients. Note that the question what is the least such class of (facet-defining) inequalities for this cone is still open.

Thus, non-negative exact games on $\mathcal{P}(N)$ form a pointed rational cone and the same is true for $\mathsf{E}_\ell(N)$. Trivial examples of non-negative exact games are indicators of supersets for singletons in N, that is, functions $\upsilon^{\uparrow\{i\}}$ for $i \in N$, where $\upsilon^{\uparrow\{i\}}(S) = 1$ if $i \in S$ and $\upsilon^{\uparrow\{i\}}(S) = 0$ otherwise. It is immediate that any non-negative game can be written uniquely as the sum of an ℓ-standardized game and of a conic combination of these trivial examples of exact games. We leave it to the reader as an easy exercise to observe that the non-negative game is exact iff its ℓ-standardized version is: adding $\upsilon^{\uparrow\{i\}}$ to μ corresponds to a shift of $C(\mu)$. These facts imply that the question of testing the extremity in the cone of non-negative exact games reduces to testing the extremity in $\mathsf{E}_\ell(N)$. Moreover, they also imply that the dimension of $\mathsf{E}_\ell(N)$ is $2^{|N|} - |N| - 1$ while the cones of exact games and the cone of non-negative exact games have both the dimension $2^{|N|} - 1$.

Definition 2 (extreme exact game)**.**
An ℓ-standardized exact game $\mu : \mathcal{P}(N) \to \mathbb{R}$ is *extreme* if it generates an extreme ray of $\mathsf{E}_\ell(N)$.

In particular, the zero game is not considered to be extreme. The fact that $\mathsf{E}_\ell(N)$ is a rational cone implies that any *extreme* ℓ-standardized exact game is a multiple of an integer-valued function $\mu : \mathcal{P}(N) \to \mathbb{Z}$. Thus, when testing the extremity of an exact game one can limit oneself to integer-valued functions. It has already been mentioned that any supermodular game is exact. Since supermodular games also form a rational polyhedral cone the set of ℓ-standardized supermodular games is a (pointed) rational polyhedral sub-cone of $\mathsf{E}_\ell(N)$. Therefore, any extreme ℓ-standardized supermodular game is also a multiple of an integer-valued function.

2.2. *Interpretation of concepts in game theory and imprecise probabilities*

This is to explain the interpretation of some of the mathematical concepts from § 2.1 in different contexts. Recall that, in cooperative game theory, the set N of variables is interpreted as the set of *players* in a coalition game and its subsets are interpreted as *coalitions*. The function $\mu : \mathcal{P}(N) \to \mathbb{R}$ with $\mu(\emptyset) = 0$ is sometimes called the characteristic function of the game and N is called the grand coalition. In profit interpretation of μ, one typically assumes $\mu(S) \geq 0$ for any $S \subseteq N$, and the goal is to divide the overall profit $\mu(N)$ among players so that the strength of coalitions, measured by $\mu(S)$ for $S \subseteq N$, is taken into account. Then the core $C(\mu)$ of μ is interpreted as the set of acceptable payoff vectors $[x_i]_{i\in N}$ of the overall profit $\mu(N) = \sum_{i\in N} x_i$ to players such that each coalition S gets at least what is its strength: $\sum_{i\in S} x_i \geq \mu(S)$.

A balanced game is a game which admits at least one acceptable payoff and an exact game is a game in which all lower bounds for payoffs to the coalitions are tight. The cone of *non-negative* balanced games is then pointed and each of its rays can be represented by a function μ normalized by $\mu(N) = 1$. Alternative description of the cone is then by means of the polytope of normalized non-negative balanced games: vertices of the polytope are normalized generators of extreme rays. The same holds for the cone of non-negative exact games.

In the context of imprecise probabilities, the set N of variables is interpreted as the *sample space* for considered (discrete) probability distributions; a term *frame of discernment* is often used instead, in particular, in connection with (the theory of) belief functions. The core $C(\mu)$ of a normalized non-negative balanced game μ is then nothing but a non-empty set of probability densities (= distributions) on N with lower bounds for probabilities $\mu(S)$, $S \subseteq N$. Thus, readers familiar with the theory of imprecise probabilities can observe that the core $C(\mu)$ is nothing but the *credal set* of probabilities on N determined by μ; the set function μ is then called a *lower probability avoiding sure loss*. Readers not familiar with imprecise probabilities is referred to [20] to check details.

The analogy can be extended to exact games: a non-negative exact game μ normalized by $\mu(N) = 1$ is nothing but a *coherent lower probability*; see [20, Corollary 3.3.4]. Therefore, the vertices of the polytope of coherent lower probabilities discussed in [14] are just the normalized generators of the cone of non-negative exact games. The indicators of supersets for singletons, mentioned in § 2.1, correspond to degenerate lower probabilities and their conic combinations to *crisp lower probabilities*, which have singleton credal sets. In particular, the ℓ-standardized normalized exact games coincide with *non-crisp coherent lower probabilities* in the context of imprecise probabilities.

A stronger concept of a normalized supermodular game coincides with the concept of a *2-monotone lower probability*. In particular, the vertices of the polytope of 2-monotone lower probabilities from [3] are just the normalized generators of extreme rays of the subcone of non-negative supermodular games. More details about the correspondence of game-theoretical concepts and those from the theory of imprecise probabilities can be found in [12], where it has been explained that even a deeper game-theoretical concept of the Shapley value has its counterpart in pignistic transformation in imprecise probabilities.

The concept of a supermodular game plays an important role in yet another context of the research on probabilistic conditional independence structures [17]. In this context, the set of variables N is interpreted as a set of (finite-valued) random variables. The ℓ-standardized supermodular games then establish an important class of *structural conditional independence models* and the extremity of a game in the respective supermodular subcone means that the assigned model is a co-atom in the lattice of structural conditional independence models.

3. The concept of a min-representation

A useful property of an exact game is that it can be represented as the minimum of a finite collection of additive games. Specifically, every $x \in \mathbb{R}^N$ defines an additive game

$$\xi \in \mathbb{R}^{\mathcal{P}(N)} \text{ by the formula } \qquad \xi(S) := \sum_{i\in S} x_i \quad \text{for any } S \subseteq N,$$

and every exact game can be obtained as a set-wise minimum of a finite collection of such additive games. This leads to the following concept.

Definition 3 (core-based and regular min-representation)**.**
We say that $\mu \in \mathbb{R}^{\mathcal{P}(N)}$ has a *min-representation* (by additive functions) if there exists a non-empty finite set $\mathcal{R} \subseteq \mathbb{R}^N$ such that

$$\forall\, S \subseteq N \qquad \mu(S) = \min_{x\in\mathcal{R}} \sum_{i\in S} x_i\,. \tag{2}$$

The set $\mathcal{R}$ is then interpreted as a min-representation of μ and every $x \in \mathcal{R}$ is assigned the class of respective *tight sets*, called briefly the *tightness class*:

$$\mathcal{T}^{\mu}_{x} := \left\{ S \subseteq N \,:\, \mu(S) = \sum_{i\in S} x_i \right\}. \tag{3}$$

The *tightness structure* of $\mathcal{R}$ is then the collection of these tightness classes: $\{\mathcal{T}^{\mu}_{x} : x \in \mathcal{R}\}$. A min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of a game μ is *core-based* if

(i) $\sum_{i\in N} x_i = \mu(N)$ for any $x \in \mathcal{R}$, which is another way of saying that any vector in $\mathcal{R}$ is an element of the core. A core-based min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of a game μ is *regular* if, moreover,

(ii) for any $x \in \mathcal{R}$, the linear hull of $\{\chi_S : S \in \mathcal{T}_x^\mu\}$ is the whole space $\mathbb{R}^N$.

An equivalent formulation of the regularity condition (ii) is that the only vector in $\mathbb{R}^N$ orthogonal to all vectors from $\{\chi_S : S \in \mathcal{T}_x^\mu\}$ is the zero vector. There exists at least one regular min-representation for every exact game.

Proposition 1 (min-representations of exact games)**.**
A game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$ is exact iff it has a core-based min-representation $\mathcal{R}$. Every exact game has a regular min-representation given by the list of all vertices of its core: $\overline{\mathcal{R}} = \mathrm{ext}\, C(\mu)$. A min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of an exact game μ is regular iff $\mathcal{R} \subseteq \mathrm{ext}\, C(\mu)$.

The proof of Proposition 1 is shifted to Appendix A.1. It follows from Proposition 1 that any exact game has the largest regular min-representation, which motivates the following definition.

Definition 4 (standard min-representation)**.**
By the *standard min-representation* of an exact game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$ will be meant the complete list $\overline{\mathcal{R}}$ of vertices of the core $C(\mu)$.

Remark. A game which admits a min-representation need not be exact as the following example shows. Put $N = \{a, b, c\}$ and consider $\mathcal{R}$ consisting of four vectors: $(2, 2, 0)$, $(2, 0, 2)$, $(0, 2, 2)$ and $(1, 1, 1)$. The formula (2) defines a game μ with $\mu(N) = 3$, $\mu(T) = 2$ for $T \subseteq N$, $|T| = 2$, and $\mu(T) = 0$ otherwise. Thus, μ has a min-representation; it is a balanced game and its core consists of a single vector $(1, 1, 1)$. Nevertheless, for any singleton $S \subseteq N$, $|S| = 1$, there is no $x \in C(\mu)$ with $\sum_{i \in S} x_i = 0 \equiv \mu(S)$; thus, μ is not exact. Taking $\mathcal{R}' = \{(1, 1, 1)\}$ leads to a min-representation of an exact game μ' given by $\mu'(S) = |S|$ for any $S \subseteq N$. Note that by [16, Theorem 1.9 in chapter 5] a game μ has a min-representation iff it is *totally balanced*, which means that, for every $\emptyset \neq T \subseteq N$, the restriction of μ to $\mathcal{P}(T)$ is a balanced game.

4. Comparison of min-representations

One can have several core-based min-representations of a given exact game. Nevertheless, there are some important relations between them.

Definition 5 (refinement ordering between tightness structures)**.**
Let μ be an exact game. We will say that a core-based min-representation $\mathcal{R}$ of μ has a *pruned tightness structure* $\{\mathcal{T}_x^\mu : x \in \mathcal{R}\}$, or briefly that $\mathcal{R}$ is *pruned*, if no pair of distinct vectors $x, y \in \mathcal{R}$ in the min-representation exists such that the inclusion $\mathcal{T}_x^\mu \subseteq \mathcal{T}_y^\mu$ holds.
Two pruned core-based min-representations $\mathcal{R}$ and $\mathcal{L}$ of μ will be *equivalent* if their tightness structures coincide: $\{\mathcal{T}_x^\mu : x \in \mathcal{R}\} = \{\mathcal{T}_y^\mu : y \in \mathcal{L}\}$. Non-equivalent (pruned core-based) min-representations $\mathcal{R}$ and $\mathcal{L}$ can be compared with the help of their induced tightness structures. We will say that the tightness structure of $\mathcal{R}$ *refines* the tightness structure of $\mathcal{L}$ if

$$\forall\, x \in \mathcal{R} \quad \exists\, y \in \mathcal{L} \qquad \mathcal{T}_x^\mu \subseteq \mathcal{T}_y^\mu .$$

Alternatively, we can say that $\{\mathcal{T}_y^\mu : y \in \mathcal{L}\}$ *coarsens* $\{\mathcal{T}_x^\mu : x \in \mathcal{R}\}$ instead. Finally, we say that two core-based min-representations are *equally fine* if their tightness structures refine each other.

It is easy to see that every class of equally fine min-representations can be represented by uniquely determined pruned tightness structure. Indeed, one can simply remove from a non-pruned $\mathcal{R}'$ those vectors whose tightness classes are contained in other tightness classes and get a pruned min-representation $\mathcal{R}$. Nevertheless, one may have several equivalent min-representations having that pruned tightness structure.

Example 1. Take $N = \{a, b, c\}$ and put

$$\mu(N) = 8, \quad \mu(\{b, c\}) = 6, \quad \mu(\{a, b\}) = \mu(\{a, c\}) = 1,$$

	(x_a, x_b, x_c)	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	N
u	$(0,1,7)$	$\bullet$	$\bullet$			$\bullet$			$\bullet$
v	$(1,7,0)$	$\bullet$			$\bullet$		$\bullet$		$\bullet$
w	$(2,0,6)$	$\bullet$		$\bullet$				$\bullet$	$\bullet$
x	$(0,4,4)$	$\bullet$	$\bullet$						$\bullet$
y	$(2,3,3)$	$\bullet$						$\bullet$	$\bullet$
z	$(0,7,1)$	$\bullet$	$\bullet$				$\bullet$		$\bullet$

Table 1: Tightness table for vectors from Example 1.

and $\mu(S) = 0$ for remaining $S \subseteq N$. This defines a supermodular game μ, which is, therefore, exact. To give examples of its min-representations consider the following sets of vectors in $\mathbb{R}^N$:

$$\mathcal{R} \;=\; \{\,\underbrace{(0,1,7)}_{u},\;\underbrace{(1,7,0)}_{v},\;\underbrace{(2,0,6)}_{w}\,\}$$

and its extensions

$$\mathcal{R}_1 = \mathcal{R} \cup \{\,\underbrace{(0,4,4)}_{x}\,\}, \qquad \mathcal{R}_2 = \mathcal{R} \cup \{\,\underbrace{(2,3,3)}_{y}\,\}, \qquad \mathcal{R}_3 = \mathcal{R} \cup \{\,\underbrace{(0,7,1)}_{z}\,\}.$$

The reader can easily verify that all these sets define core-based min-representations of μ and the respective tightness classes are encoded in Table 1. Then $\mathcal{R}_1$ has not a pruned tightness structure because $\mathcal{T}^{\mu}_x \subseteq \mathcal{T}^{\mu}_u$. Analogously, $\mathcal{T}^{\mu}_y \subseteq \mathcal{T}^{\mu}_w$ implies that $\mathcal{R}_2$ is not pruned. The min-representations $\mathcal{R}_1$ and $\mathcal{R}_2$ are equally fine and the corresponding equivalence class can be represented by pruned min-representation $\mathcal{R}$. Clearly, $\mathcal{R}$ refines trivially its superset $\mathcal{R}_3$; on the other hand, $\mathcal{R}_3$ does not refine $\mathcal{R}$ because $\mathcal{T}^{\mu}_z$ is not contained in any of $\mathcal{T}^{\mu}_u$, $\mathcal{T}^{\mu}_v$ and $\mathcal{T}^{\mu}_w$.

The point is that the set of tightness structures of min-representations of a fixed exact game forms a lattice.

Proposition 2 (lattice of tightness structures)**.**
Given an exact game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$, the collection of tightness structures of its core-based min-representations is a lattice with respect to the refinement relation and the equivalence of being equally fine. Formally, elements of the lattice are the equivalence classes ordered by the refinement relation. The largest element in the lattice is the *coarsest tightness structure* of the standard min-representation. The *finest tightness structure* can be constructed from the above-mentioned standard min-representation $\overline{\mathcal{R}} = \mathsf{ext}\, C(\mu)$ as follows:

[i] for any $S \subseteq N$ consider the set $\overline{\mathcal{R}}[S] := \{x \in \overline{\mathcal{R}} \,:\, S \in \mathcal{T}^{\mu}_x\}$ of vertices of $C(\mu)$ having S in the tightness class and define $y^S \in C(\mu)$ as their arithmetic mean:

$$y^S \;:=\; \sum_{x\in\overline{\mathcal{R}}[S]} \frac{1}{|\overline{\mathcal{R}}[S]|}\cdot x\,; \qquad \text{note that } \mathcal{T}^{\mu}_y = \bigcap_{x\in\overline{\mathcal{R}}[S]} \mathcal{T}^{\mu}_x \quad \text{for } y = y^S\,.$$

[ii] prune (the tightness structure of) the min-representation $\{y^S \,:\, S \subseteq N\}$ of μ by the removal of ($y^S = y$ with) non-maximal classes $\mathcal{T}^{\mu}_y$ with respect to inclusion.

The proof of Proposition 2 is shifted to Appendix A.2. Note that one can extend the proof to observe that other min-representations of μ with the finest tightness structure can be obtained by taking different positive convex combinations of vectors in $\overline{\mathcal{R}}[S]$; they are, however, equivalent to the given one in [ii] in sense of Definition 5.

Definition 6 (finest min-representation)**.**
Any core-based min-representation of an exact game μ which yields the pruned finest tightness structure will be named its *finest min-representation*.

	(x_a, x_b, x_c)	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	N
x	$(0,1,7)$	•	•			•			•
y	$(0,7,1)$	•	•				•		•
z	$(1,7,0)$	•			•		•		•
u	$(2,6,0)$	•			•			•	•
v	$(2,0,6)$	•		•				•	•
w	$(1,0,7)$	•		•		•			•

Table 2: Tightness table for the standard min-representation from Example 2.

	(x_a, x_b, x_c)	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	N
e	$(0,4,4)$	•	•						•
f	$(\frac{3}{2},0,\frac{13}{2})$	•		•					•
g	$(\frac{3}{2},\frac{13}{2},0)$	•			•				•
h	$(\frac{1}{2},\frac{1}{2},7)$	•				•			•
s	$(\frac{1}{2},7,\frac{1}{2})$	•					•		•
t	$(2,3,3)$	•						•	•

Table 3: Tightness table for a finest min-representation from Example 2.

The procedure from Proposition 2 is illustrated by the following example.

Example 2. Consider the exact game μ over $N = \{a,b,c\}$ from Example 1. The core of μ has six vertices, given in Table 2, where the respective tightness classes are shown. To illustrate the procedure described in Proposition 2 consider $S = \{a\}$. Then $\overline{\mathcal{R}}[S] = \{x,y\}$ and their arithmetic mean is

$$\frac{1}{2}\cdot(0,1,7) + \frac{1}{2}\cdot(0,7,1) = (0,4,4)\,.$$

Similarly, if $S = \{b,c\}$ then one has $\overline{\mathcal{R}}[S] = \{u,v\}$ and the arithmetic mean is

$$\frac{1}{2}\cdot(2,6,0) + \frac{1}{2}\cdot(2,0,6) = (2,3,3)\,.$$

In case $S = \emptyset$ and $S = N$ one has $\overline{\mathcal{R}}[S] = \overline{\mathcal{R}}$ and the arithmetic mean is $(1,\frac{7}{2},\frac{7}{2})$ and the respective tightness class is just $\{\emptyset, N\}$. Therefore, the vector $(1,\frac{7}{2},\frac{7}{2})$ is removed during the pruning step [ii]. The overall result of the procedure from Proposition 2 is shown in Table 3.

5. Extremity characterization

If $\mu \in \mathsf{E}_\ell(N)$ is an ℓ-standardized exact game, then its core $C(\mu)$ consists of non-negative vectors and any core-based min-representation of μ is a non-empty finite subset of $[0,\infty)^N$.

5.1. Some arrangement

To formalize our result consider a core-based min-representation $\mathcal{R}$ of μ. Let us choose and fix an auxiliary index set Γ for the vectors in $\mathcal{R}$ and imagine them being arranged in the form of a *real array* $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ whose rows are indexed by Γ and columns by N:

$$\mathsf{x} \;:=\; [\,\mathsf{x}(\gamma,i)\,]_{\gamma\in\Gamma, i\in N} \in \mathbb{R}^{\Gamma\times N} \quad \text{where} \;\; \mathcal{R} = \{\,[\mathsf{x}(\gamma,i)]_{i\in N} :\; \gamma\in\Gamma\,\} \subseteq C(\mu).$$

The minimization formula (2) for $\mathcal{R}$ means that μ is obtained by a set-wise minimization in the array x over its rows:

$$\forall\, S \subseteq N \qquad \mu(S) = \min_{\gamma\in\Gamma} \sum_{i\in S} \mathsf{x}(\gamma, i)\,. \tag{4}$$

In this context, the tightness classes (3) correspond to elements of Γ and we also ascribe a certain linear subspace of $\mathbb{R}^N$ to each of them:

$$L_\gamma \;:=\; \mathrm{Lin}\,\{\chi_S \,:\, S\in\mathcal{T}_\gamma\} \qquad \text{where } \mathcal{T}_\gamma \;:=\; \{S\subseteq N \,:\, \mu(S) = \textstyle\sum_{i\in S}\mathsf{x}(\gamma,i)\} \text{ for any } \gamma\in\Gamma. \tag{5}$$

Observe that $\dim(L_\gamma)\geq 1$ as $N\in\mathcal{T}_\gamma$ for all $\gamma\in\Gamma$ by the core-based condition (i).

5.2. The linear equation system

Every core-based min-representation satisfying (4) is ascribed a system of linear constraints for an unknown array

$$\mathsf{y} \;=\; [\,\mathsf{y}(\gamma,i)\,]_{\gamma\in\Gamma, i\in N} \in \mathbb{R}^{\Gamma\times N}\,.$$

The constraints are determined by the *tightness* classes $\mathcal{T}_\gamma$ for $\gamma\in\Gamma$:

(a) $\forall\,\gamma\in\Gamma\;\; \forall\, i\in N$ with $\{i\}\in\mathcal{T}_\gamma \qquad \mathsf{y}(\gamma,i)=0,$

(b) $\forall\, S\subseteq N,\ |S|\geq 2,\ \ \forall\,\gamma,\rho\in\Gamma$ with $S\in\mathcal{T}_\gamma\cap\mathcal{T}_\rho \qquad \sum_{i\in S}\mathsf{y}(\gamma,i) = \sum_{i\in S}\mathsf{y}(\rho,i).$

The meaning of the equivalence of pruned core-based min-representations from Definition 5 is that they give rise to the same linear equation system (a)-(b) up to re-ordering of rows; indeed, the system is evidently given by the tightness structure.

It is easy to observe that the starting array $\mathsf{x}\in\mathbb{R}^{\Gamma\times N}$ satisfies (a)-(b); the reason is that the constraints are determined by x through the tightness classes and μ is ℓ-standardized. On the other hand, additional "unwanted" solutions to (a)-(b) may exist. Whenever, for some $\gamma^*\in\Gamma$, a vector $v\in\mathbb{R}^N$ belongs to the orthogonal complement of L_{γ^*} in $\mathbb{R}^N$, that is, whenever $\langle\chi_S, v\rangle = 0$ for any $S\in\mathcal{T}_{\gamma^*}$, then the formula

$$\mathsf{y}(\gamma,i) \;:=\begin{cases} v_i & \text{if } \gamma=\gamma^*,\\ 0 & \text{if } \gamma\neq\gamma^*,\end{cases} \qquad \text{for any } \gamma\in\Gamma,\ i\in N, \tag{6}$$

also gives a solution to (a)-(b). The linear hull of such additional solutions is characterized as the kernel of a certain linear mapping.

Proposition 3. Given $\mathsf{z}\in\mathbb{R}^{\Gamma\times N}$ satisfying (a)-(b) there exists a uniquely determined ℓ-standardized game $v\in\mathbb{R}^{\mathcal{P}(N)}$ such that

$$\forall\,\gamma\in\Gamma\;\;\forall\, S\in\mathcal{T}_\gamma \qquad v(S) = \sum_{i\in S}\mathsf{z}(\gamma,i)\,. \tag{7}$$

The mapping $\mathcal{A}: \mathsf{z}\in R_\Gamma\subseteq\mathbb{R}^{\Gamma\times N}\mapsto v\in\mathbb{R}^{\mathcal{P}(N)}$, where R_Γ is the space of solutions to (a)-(b), is linear. The linear space $\mathcal{A}_{-1}(\mathbf{0})$ is the linear hull of solutions to (a)-(b) of type (6) and has the dimension

$$\sum_{\gamma\in\Gamma}\mathrm{codim}\,(L_\gamma) \qquad \text{where, for any } \gamma\in\Gamma,\ \mathrm{codim}\,(L_\gamma) := |N| - \dim(L_\gamma) = \dim(L_\gamma^\perp)$$

denotes the dimension of the orthogonal complement $L_\gamma^\perp$ of L_γ in $\mathbb{R}^N$.

The uniqueness of the game v from (7) follows from the fact $\mathcal{P}(N) = \bigcup_{\gamma\in\Gamma}\mathcal{T}_\gamma$, which itself follows from the exactness of μ. Indeed, for any $S\subseteq N$ some $\gamma\in\Gamma$ exists with $S\in\mathcal{T}_\gamma$ and the formula (7) defines the value of $v(S)$. The proof of Proposition 3 is shifted to Appendix A.3. A substantial value in our extremity criterion is the so-called essential dimension of the space of solutions to (a)-(b); basically, the solutions representing the zero game are ignored.

Definition 7 (essential dimension)**.**
Let $\mu \in \mathsf{E}_\ell(N)$ be an ℓ-standardized exact game. Assume that $\mathcal{R}$ is a core-based min-representation of μ, arranged in an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ satisfying (4). The *essential dimension* of $\mathcal{R}$ is the difference of the dimension of the linear space R_Γ of solutions to (a)-(b) and that of the space $\mathcal{A}_{-1}(\mathbf{0})$ from Proposition 3.

Let us call a solution $\mathsf{y} \in R_\Gamma$ to (a)-(b) *pure* if it belongs to the orthogonal complement of $\mathcal{A}_{-1}(\mathbf{0})$ in R_Γ. Thus, the essential dimension is the dimension of the space T_Γ of pure solutions. Nevertheless, the next example shows that it may happen that no pure solution to (a)-(b) is a min-representation (of μ). The example also illustrates the concepts introduced in § 5.

Example 3. Consider the exact game μ over $N = \{a, b, c\}$ from Example 1:

$$\mu(N) = 8, \quad \mu(\{b, c\}) = 6, \quad \mu(\{a, b\}) = \mu(\{a, c\}) = 1,$$

and $\mu(S) = 0$ for remaining $S \subseteq N$. In Example 2, a finest representation of μ was constructed, shown in Table 3. In this case $\Gamma = \{e, f, g, h, s, t\}$ and the $\Gamma \times N$-array has 18 entries. The condition (a) gives rise to 3 equality constraints in that array. The condition of type (b) gives 5 independent constraints of type $\sum_{i\in N} \mathsf{y}(e, i) = \sum_{i\in N} \mathsf{y}(\gamma, i)$ for $\gamma \in \Gamma \setminus \{e\}$. In particular, the dimension of the space R_Γ of solutions to (a)-(b) is 18-3-5=10.

As concerns the linear spaces from (5) and their dimensions, consider, for example, the f-row in the array. Then one has

$$L_f = \mathrm{Lin}\{\,\mathsf{0}, \chi_{\{b\}}, \chi_N\} = \{(r, q, r) \;:\; r, q \in \mathbb{R}\},$$

and $\dim(L_f) = 2$. The reader can analogously observe on basis of Table 3 that $\dim(L_\gamma) = 2$ for any $\gamma \in \Gamma$; therefore, $\mathrm{codim}\,(L_\gamma) = 1$ for all $\gamma \in \Gamma$. In particular, the space of dummy solutions $\mathcal{A}_{-1}(\mathbf{0})$ has the dimension 6 and the essential dimension is 10-6=4.

To observe that no pure solution to (a)-(b) yields a min-representation of μ consider again the space L_f. Assume that $\mathsf{y} \in T_\Gamma$ is a pure solution to (a)-(b). Then its f-row $\mathsf{y}(f, *) \in \mathbb{R}^N$ must be perpendicular to any $v \in L_f^\perp$ because any such v gives rise to a solution in $\mathcal{A}_{-1}(\mathbf{0})$ of the form (6). This means that $\mathsf{y}(f, *)$ has to belong to $(L_f^\perp)^\perp = L_f = \{(r, q, r) \;:\; r, q \in \mathbb{R}\}$. Since y satisfies (a) and $\{b\} \in \mathcal{T}_f^\mu$ one has $\mathsf{y}(f, b) = 0$, implying that $\mathsf{y}(f, *) = (r, 0, r)$ for some $r \in \mathbb{R}$. Assume, moreover, that y yields a core-based min-representation of μ, which implies that any row of y belongs to $C(\mu)$. Thus, $\mathsf{y}(f, *) \in C(\mu)$, which contradicts the fact that there is no vector of the form $(r, 0, r)$ in $C(\mu)$. Indeed, the reader can consult Table 2 to observe that elements $(x_a, x_b, x_c) \in C(\mu)$ with $x_b = 0$ have the form $(r, 0, 8 - r)$ for $1 \le r \le 2$, implying $x_a \neq x_c$.

5.3. The very criterion

The crucial observation is as follows.

Proposition 4. A non-zero ℓ-standardized exact game $\mu \in \mathsf{E}_\ell(N)$ is *extreme* in $\mathsf{E}_\ell(N)$ iff, for each core-based min-representation of μ, arranged in the form of an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ satisfying (4), the essential dimension is 1.

The proof of Proposition 4 is shifted to Appendix A.4. Nevertheless, one can avoid testing all core-based min-representations. The reason is that the essential dimension is monotone with respect to the refinement ordering.

Proposition 5. Let $\mathcal{R}$ and $\mathcal{L}$ be core-based min-representations of a non-zero exact ℓ-standardized game μ such that the tightness structure of $\mathcal{R}$ refines the tightness structure of $\mathcal{L}$. Then one has

$$1 \le \mathrm{esdim}\,(\mathcal{L}) \le \mathrm{esdim}\,(\mathcal{R}) \le 2^{|N|} - |N| - 1\,.$$

Thus, the essential dimension of equally fine min-representations is the same.

The proof of Proposition 5 is shifted to Appendix A.5. It allows us to simplify the extremity criterion because testing of all core-based min-representations can be replaced by testing solely one of the finest min-representations.

Corollary 1. A non-zero ℓ-standardized exact game μ is *extreme* in $\mathsf{E}_\ell(N)$ iff the essential dimension of its (arbitrary) finest min-representation is 1.

	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	N
$(2,2,0,0)$	•			•	•						•			•	•	•
$(2,1,1,0)$	•				•					•					•	•
$(2,1,0,1)$	•			•					•						•	•
$(1,2,1,0)$	•				•			•						•		•
$(1,2,0,1)$	•			•			•							•		•
$(1,1,2,0)$	•				•			•		•			•			•
$(1,0,2,1)$	•		•			•				•			•			•
$(1,0,1,2)$	•		•			•			•			•				•
$(0,2,1,1)$	•	•					•	•						•		•
$(0,1,1,2)$	•	•				•	•					•				•
$(2,0,1,1)$	•		•						•	•					•	•
$(1,1,0,2)$	•			•			•		•			•				•
$(0,1,2,1)$	•	•				•		•					•			•

Table 4: Tightness table for the standard min-representation from Example 4.

Proof. Combine Proposition 4, Proposition 5 and Proposition 2. □

One can use Corollary 1 in combination with the construction in Proposition 2, which allows one to obtain one of the finest core-based min-representations on basis of the vertices of the core of μ. Another note is that the arguments in the proof of Proposition 5 and a consideration extending the one in the necessity proof for Proposition 4 allow one to show that esdim $(\mathcal{R})$ for any finest min-representation $\mathcal{R}$ of μ is the dimension of the least face of $\mathsf{E}_\ell(N)$ containing μ. The monotonicity of the essential dimension is illustrated by the next example.

Example 4. Put $N = \{a,b,c,d\}$ and introduce a game over N as follows:

$$\mu(N) = 4, \qquad \mu(S) = 2 \quad \text{for } S \subseteq N \text{ with } |S| = 3,$$
$$\mu(S) = 1 \quad \text{for any } S \subseteq N \text{ with } |S| = 2 \text{ except for } S = \{c,d\},$$
$$\mu(S) = 0 \quad \text{for remaining } S \subseteq N.$$

This is a supermodular game and, therefore, an exact game on $\mathcal{P}(N)$. The core $\overline{\mathcal{R}} = \mathsf{ext}\, C(\mu)$ consists of 13 vertices. The respective tightness array is shown in Table 4. Every solution to (a)-(b) for the standard min-representation $\overline{\mathcal{R}}$ appears to be a multiple of the starting array shown in the very left column of the table. Thus, one has $\dim(R_\Gamma) = 1$ and, since codim $(L_\gamma) = 0$ for any $\gamma \in \Gamma$, one has esdim $(\overline{\mathcal{R}}) = 1$.

The first 10 rows in Table 4 also define a min-representation $\mathcal{R}'$ of μ, which, by definition, refines $\overline{\mathcal{R}}$. We found out by computation that esdim $(\mathcal{R}') = 4$. In particular, the condition from Proposition 4 allows one to conclude that μ is not extreme in $\mathsf{E}_\ell(N)$.

The procedure from Proposition 2 leads to a finest min-representation $\mathcal{R}$ with 10 vectors shown in Table 5. One can compute its essential dimension esdim $(\mathcal{R}) = 9$, which illustrates the inequality from Proposition 5:

$$1 \leq \underbrace{\text{esdim}\,(\overline{\mathcal{R}})}_{=1} \leq \underbrace{\text{esdim}\,(\mathcal{R}')}_{=4} \leq \underbrace{\text{esdim}\,(\mathcal{R})}_{=9} \leq 11 \equiv 2^4 - 4 - 1\,.$$

This basically means that μ does not belong to the relative interior of $\mathsf{E}_\ell(N)$ but to its face of codimension $2 = 11 - 9$, sometimes named a *ridge* of $\mathsf{E}_\ell(N)$.

We have already mentioned that μ is not extreme in $\mathsf{E}_\ell(N)$. To illustrate this realize that it can be written as the sum of two exact games, namely of

$$\mu^1(N) = 2, \quad \mu^1(S) = 1 \ \text{ for } S \subseteq N, |S| = 3, \quad \mu^1(\{a,c\}) = \mu^1(\{b,c\}) = \mu^1(\{b,d\}) = 1\,,$$

and $\mu^1(S) = 0$ for remaining $S \subseteq N$, whose core has three vertices, namely

$$\mathsf{ext}\, C(\mu^1): \qquad (1,1,0,0), \quad (0,1,1,0), \quad (0,0,1,1),$$

and of the game

$$\mu^2(N) = 2, \ \ \mu^2(S) = 1 \quad \text{for } S \subseteq N, |S| = 3, \quad \mu^2(\{a,b\}) = \mu^2(\{a,d\}) = 1\,,$$

	∅	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	N
$(2,2,0,0)$	•			•	•						•			•	•	•
$(0,\frac{4}{3},\frac{4}{3},\frac{4}{3})$	•	•														•
$(\frac{4}{3},0,\frac{4}{3},\frac{4}{3})$	•		•													•
$(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})$	•					•										•
$(\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{3}{2})$	•						•									•
$(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2})$	•							•								•
$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$	•								•							•
$(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})$	•									•						•
$(\frac{2}{3},\frac{2}{3},\frac{2}{3},2)$	•											•				•
$(\frac{2}{3},\frac{2}{3},2,\frac{2}{3})$	•												•			•

Table 5: Tightness table for a finest min-representation from Example 4.

and $\mu^2(S) = 0$ for remaining $S \subseteq N$, whose core has four vertices, namely

$$\mathsf{ext}\, C(\mu^2): \qquad (1,1,0,0), \quad (1,0,1,0), \quad (1,0,0,1), \quad (0,1,0,1)\,.$$

The above mentioned min-representation $\mathcal{R}'$ is just the set of vertices of the Minkowski sum $C(\mu^1) \oplus C(\mu^2)$.

The fact that esdim $(\overline{\mathcal{R}}) = 1$ for the standard min-representation $\overline{\mathcal{R}}$ of μ means, by the supermodular criterion from [19], that μ is an extreme ℓ-standardized supermodular game. Actually, μ and its multiples are the only examples of extreme ℓ-standardized supermodular games over 4 variables which are not extreme in the larger exact cone; this observation was a result of our detailed analysis of the case $|N| \leq 4$. The last note is that $\overline{\mathcal{R}}$ is not the only regular min-representation which has the essential dimension 1. We found out by computation that the least such regular min-representation of μ exists and has 9 vectors: these are all the vectors from Table 4 except $(2,1,1,0)$, $(2,1,0,1)$, $(1,2,1,0)$ and $(1,2,0,1)$.

5.4. Remarks on implementation

Consider the arrangement of a core-based min-representation $\mathcal{R}$ of an exact game μ described in § 5.1. For computational and implementation reasons, it is useful to consider a big zero-one *tightness array* encoding the tightness structure. This array ι has rows indexed by Γ and columns by subsets of N:

$$\iota \;:=\; [\,\iota(\gamma,S)\,]_{\gamma\in\Gamma,S\subseteq N} \in \{0,1\}^{\Gamma\times\mathcal{P}(N)} \quad \text{where} \quad \iota(\gamma,S) = \begin{cases} 1 & \text{if } \mu(S) = \sum_{i\in S} \mathsf{x}(\gamma,i), \\ 0 & \text{otherwise.} \end{cases}$$

To test whether (4) holds one can consider the tightness array $\iota \in \{0,1\}^{\Gamma\times\mathcal{P}(N)}$ and check whether each column in ι contains least once 1. Tables 2-5 offer pictorial representations of respective tightness arrays; just bullets encode 1's and blank spaces encode 0's.

The constraints (a)-(b) from § 5.2 for fixed Γ can be written in the form of a matrix equality $\mathbf{C}_\Gamma \cdot y = \mathbf{0}$, where $\mathbf{C}_\Gamma$ is an appropriate *constraint matrix* with entries in $\{-1,0,+1\}$. The rows of $\mathbf{C}_\Gamma$ encode the constraints and its columns correspond to elements of $\Gamma \times N$. The matrix is sparse: every constraint of type (a) is encoded by a row with one non-zero component while any constraint of type (b) for $S \subseteq N$, $|S| \geq 2$, is encoded by a row containing $|S|$-times a component $+1$, $|S|$-times a component -1 and 0 otherwise.

The number of rows in $\mathbf{C}_\Gamma$ can be reduced because some of the constraints of type (b) follow from the others. For example, whenever $S \subseteq N$, $|S| \geq 2$, belongs to $\mathcal{T}_\gamma \cap \mathcal{T}_\rho \cap \mathcal{T}_\sigma$ for different $\gamma,\rho,\sigma \in \Gamma$ then only two constraints

$$\sum_{i\in S} \mathsf{y}(\gamma,i) - \sum_{i\in S} \mathsf{y}(\rho,i) = 0 \quad \text{and} \quad \sum_{i\in S} \mathsf{y}(\gamma,i) - \sum_{i\in S} \mathsf{y}(\sigma,i) = 0$$

are enough. Therefore, if we denote, for $S \subseteq N$, by $\lambda(S)$ the number of 1's in the respective column of the tightness array ι, then the maximally reduced number of rows of $\mathbf{C}_\Gamma$ is

$$\sum_{S\subseteq N,|S|=1} \lambda(S) + \sum_{S\subseteq N,|S|\geq 2} [\lambda(S) - 1]\,.$$

Testing of the condition from Proposition 4 for fixed Γ can be realized by computing the *nullity* null $(\mathbf{C}_\Gamma)$ of the matrix $\mathbf{C}_\Gamma$, which is the dimension of the space of solution to $\mathbf{C}_\Gamma \cdot y = \mathbf{0}$. As concerns the essential dimension, it follows from results in § 5.2 that it can be computed by the formula

$$\mathrm{esdim}\,(\mathcal{R}) \;=\; \mathrm{null}\,(\mathbf{C}_\Gamma) - \sum_{\gamma\in\Gamma} \mathrm{codim}\,(L_\gamma)\,.$$

Finally, computing codim (L_γ) for a given $\gamma \in \Gamma$ can be realized as follows. The respective γ-row of the tightness array ι allows one to get the list $\mathcal{T}_\gamma$ of tight sets for γ. Thus, one can compose an auxiliary matrix whose rows are indicators $\chi_S \in \mathbb{R}^N$ for $S \in \mathcal{T}_\gamma$. Then the rank of this matrix is nothing but the dimension of L_γ and, by the rank-nullity theorem, the nullity of the matrix is $|N| - \dim(L_\gamma) = \dim(L_\gamma^\perp) = \mathrm{codim}\,(L_\gamma)$.

The computing routines were implemented in R language [15] and the web interface was created using shiny framework for R [4]. To compute the vertices of the core of a given game, we have used rcdd package [8] for generating extreme points and rays of a general convex polyhedron specified by linear inequalities. The package itself is based on double description method [7]. Bitmaps were used to speed up the operations with systems of subsets and to handle tightness structures.

6. On uniqueness of min-representations

In general, one can have several min-representations of an exact game. But it may also be the case that only one core-based min-representation exists, which happens if and only if the following condition holds.

Definition 8 (oxytrophic game)**.**
We say that an exact game $\mu : \mathcal{P}(N) \to \mathbb{R}$ is *oxytrophic* if $\forall\, x \in \mathsf{ext}\, C(\mu)$

$$\exists\, S \subseteq N \text{ with } \mu(S) = \textstyle\sum_{i\in S} x_i \;:\; \forall\, y \in \mathsf{ext}\, C(\mu),\; y \neq x \quad \mu(S) < \displaystyle\sum_{i\in S} y_i\,. \tag{8}$$

Realize that (8) means that x is the only vector $z \in C(\mu)$ with $S \in \mathcal{T}^\mu_z$. Thus, the condition from Definition 8 requires that each vector in the standard min-representation is involved in any core-based min-representation. This relevant mathematical concept has already appeared in the literature and we simply took over the terminology introduced by Rosenmüller [16, § 3 of chapter 5]. The following gives an example of an oxytrophic game, which is extreme in $\mathsf{E}_\ell(N)$.

Example 5. Put $N = \{a, b, c, d\}$ and consider $\mathcal{R} \subseteq \mathbb{R}^N$ consisting of 4 vectors (x_a, x_b, x_c, x_d), namely $(1, 1, 1, 1)$, $(2, 2, 0, 0)$, $(2, 0, 2, 0)$, $(0, 2, 2, 0)$. Then the minimization formula (2) gives

$$\mu(N) = 4,\;\; \mu(\{a, b, c\}) = 3,\;\; \mu(\{a, b, d\}) = \mu(\{a, c, d\}) = \mu(\{b, c, d\}) = 2$$
$$\mu(\{a, b\}) = \mu(\{a, c\}) = \mu(\{b, c\}) = 2, \quad \mu(S) = 0 \;\text{ for other } S \subseteq N.$$

One can easily verify by computation that $\mathcal{R} = \mathsf{ext}\, C(\mu)$, which allows one to check the condition (8) for arbitrary $x \in \mathsf{ext}\, C(\mu)$: $(1, 1, 1, 1)$ has one respective set $S = \{a, b, c\}$, while $(2, 2, 0, 0)$ has even two respective sets $S = \{c\}$ and $S = \{c, d\}$, etc. In particular, μ is oxytrophic. Moreover, μ is also an example of an (extreme) exact game which is not supermodular:

$$\mu(\{a, c\}) + \mu(\{b, c\}) = 4 > 3 = \mu(\{a, b, c\}) + \mu(\{c\})\,.$$

An interesting observation is that, in case $|N| = 3$, the ℓ-standardized oxytrophic games are just the zero game and extreme exact games. However, in case $|N| = 4$, an extreme exact game exists which is not oxytrophic. The next example is even a supermodular game.

Example 6. Put $N = \{a, b, c, d\}$ and introduce

$$\mu(N) = 2, \quad \mu(\{a, b, c\}) = \mu(\{a, b, d\}) = \mu(\{a, c, d\}) = 1,$$

	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	N
$(2,0,1,0)$	•		•		•	•			•	•			•		•	•
$(0,2,1,0)$	•	•			•	•	•	•					•	•		•
$(1,1,0,1)$	•			•		•	•		•			•				•
$(\frac{3}{2},\frac{3}{2},0,0)$	•			•	•						•					•

Table 6: Tightness table for the finest min-representation from Example 7.

and $\mu(S)=0$ for other $S\subseteq N$. The core $C(\mu)$ has 7 vertices (x_a,x_b,x_c,x_d), namely four substantial ones denoted by

$$\mathcal{R}: \qquad (2,0,0,0), \quad (0,1,1,0), \quad (0,1,0,1), \quad (0,0,1,1),$$

and three additional ones

$$(1,1,0,0), \quad (1,0,1,0), \quad (1,0,0,1).$$

The vectors in $\mathcal{R}$ satisfy (8): $S=\{b,c,d\}$ for $(2,0,0,0)$, $S=\{a,d\}$ for $(0,1,1,0)$, $S=\{a,c\}$ for $(0,1,0,1)$ and $S=\{a,b\}$ for $(0,0,1,1)$. However, the remaining 3 vertices of $C(\mu)$ do not satisfy (8). Therefore, μ is not oxytrophic.

The condition (8) for the vectors in $\mathcal{R}$ implies that every core-based min-representation involves $\mathcal{R}$. Nevertheless, the vectors in $\mathcal{R}$ already provide a (regular) min-representation of μ. Therefore, the procedure from Proposition 2 for constructing the finest tightness structure results in $\mathcal{R}$, which is the unique finest min-representation in this case.

On the other hand, an extreme exact (non-oxytrophic) game can have several inclusion-minimal regular min-representations as the following example shows.

Example 7. Put $N=\{a,b,c,d\}$ and

$$\mu(N)=3, \quad \mu(\{a,b,c\})=\mu(\{a,b,d\})=\mu(\{a,b\})=2, \quad \mu(\{a,c,d\})=\mu(\{b,c,d\})=\mu(\{a,c\})=\mu(\{b,c\})=1\,,$$

and $\mu(S)=0$ for remaining for $S\subseteq N$. Then core $\overline{\mathcal{R}}=\mathsf{ext}\,C(\mu)$ consists of five vertices. Three of them satisfy the oxytrophic condition (8):

$$\mathcal{L}: \quad (2,0,1,0), \quad (0,2,1,0), \quad (1,1,0,1),$$

while the remaining two vertices $(1,2,0,0)$ and $(2,1,0,0)$ do not. Indeed, one has $S=\{b,c,d\}$ or $S=\{b,d\}$ for $(2,0,1,0)$, $S=\{a\}$ for $(0,2,1,0)$ and $S=\{a,b,c\}$ for $(1,1,0,1)$. Thus, μ is not oxytrophic.

The procedure from Proposition 2 results in the finest tightness structure of a min-representation $\mathcal{R}$ shown in Table 6. One can show that $\mathrm{esdim}\,(\mathcal{R})=1$, which implies that μ is extreme in $\mathsf{E}_\ell(N)$. Note that finest min-representations are not regular. In this case, two inclusion-minimal regular min-representations exist, namely $\mathcal{L}\cup\{\,(1,2,0,0)\,\}$ and $\mathcal{L}\cup\{\,(2,1,0,0)\,\}$.

The last example shows that an oxytrophic game need not be extreme.

Example 8. Put $N=\{a,b,c,d\}$ and

$$\mu(N)=2, \quad \mu(S)=1 \ \text{ for } S\subseteq N,\, |S|=3, \quad \mu(\{a,b\})=\mu(\{c,d\})=1, \quad \text{and } \ \mu(S)=0 \ \text{ for other } S\subseteq N.$$

Then $\mathcal{R}=\mathsf{ext}\,C(\mu)$ has four vectors (x_a,x_b,x_c,x_d), namely

$$(0,1,0,1), \quad (0,1,1,0), \quad (1,0,0,1), \quad (1,0,1,0)\,.$$

The vectors in $\mathcal{R}$ satisfy (8): $S=\{a,c\}$ for $(0,1,0,1)$, $S=\{a,d\}$ for $(0,1,1,0)$, $S=\{b,c\}$ for $(1,0,0,1)$ and $S=\{b,d\}$ for $(1,0,1,0)$. Thus, μ is oxytrophic. On the other hand, μ is the sum of two other ℓ-standardized supermodular games μ^1 and μ^2, where μ^1 is the indicator of supersets of $\{a,b\}$ and μ^2 is the indicator of supersets of $\{c,d\}$.

7. Conclusions

To have a practical result of our theoretical work we prepared a web-based application for testing the extremity of an ℓ-standardized integer-valued exact game, based on the criterion from Corollary 1. It is available at

http://gogo.utia.cas.cz/finest-min-representation/ .

It also allows one to test whether the represented exact game is oxytrophic.

We have mentioned that the criterion for exact extremity is a modification of a former (simpler) criterion for supermodular extremity from [19]. Since the former supermodular criterion has not been implemented so far we decided to prepare a web-based application for testing the supermodular extremity too:

http://gogo.utia.cas.cz/standard-min-representation/ .

Moreover, we have also performed a few experiments with those routines. We have tested our criterion on 41 permutational types of 398 extreme ℓ-standardized exact games over 4 variables; these were also earlier listed by [14]. What we have found out is that 20 of these types are oxytrophic; one of them is mentioned in Example 5. The remaining types are not, but for 19 of these the least regular min-representation exists; one of them is mentioned in Example 6. We also found two types of extreme exact games for which two inclusion-minimal regular min-representations exist; one of these two types is given in Example 7.

As far as we know our Corollary 1 provides the first complete criterion for the extremity of an exact game, where the completeness means that a necessary and sufficient condition is given. The reader may think that computing the finest min-representation and its essential dimension is unnecessarily complicated and may wonder whether a simpler criterion exists. This is indeed one of natural open tasks one can raise in this context. Note that our intuitive expectation, based on geometric consideration, is that there is no (excessively) easy complete criterion for the extremity of an exact game: realize that the exact cone is high-dimensional and has the exponential number of facets and extreme rays in $|N|$.

On the other hand, we believe that our computational experiments confirmed that the criterion from Corollary 1 can be implemented efficiently. Moreover, it simplifies considerably in some special cases like in case of oxytrophic games mentioned in § 6. Further group of open tasks concerns the computational complexity of testing the extremity. One can ask whether the characterization problem in NP-complete or whether testing the condition $\mathrm{esdim}(\mathcal{R}) = 1$ is polynomial in $|\mathcal{R}|$. Questions of this kind could be a topic of further research.

Acknowledgements

We thank our colleague Tomáš Kroupa for recomputing extreme exact games over four variables. This research has been supported by the grant project of GAČR, number 16-12010S.

Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Let us recall what we are going to prove.

Proposition 1: A game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$ is exact iff it has a core-based min-representation $\mathcal{R}$.
Every exact game has a regular min-representation given by the list of all vertices of its core: $\overline{\mathcal{R}} = \mathsf{ext}\, C(\mu)$.
A min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of an exact game μ is regular iff $\mathcal{R} \subseteq \mathsf{ext}\, C(\mu)$.

Note that the proof uses special geometric concept of a *normal fan* (of a polytope) from [22].

Proof. If μ has a min-representation $\mathcal{R}$ satisfying the core-based condition

(i) $\sum_{i \in N} x_i = \mu(N)$ for any $x \in \mathcal{R}$, then the min-representability condition

$$\forall\, S \subseteq N \qquad \mu(S) = \min_{x \in \mathcal{R}} \sum_{i \in S} x_i \tag{2}$$

implies $\emptyset \neq \mathcal{R} \subseteq C(\mu)$ and the condition of exactness for μ is evident.
Conversely, given an exact game μ we put $\overline{\mathcal{R}} = \mathsf{ext}\, C(\mu)$ and use

$$\forall\, S \subseteq N \;\; \exists\, x \in \mathsf{ext}\, C(\mu) \qquad \sum_{i \in S} x_i = \mu(S) \tag{1}$$

to observe that (2) holds with $\overline{\mathcal{R}}$ in place of $\mathcal{R}$. The core-based condition (i) for $\overline{\mathcal{R}}$ is evident. To verify the regularity condition

(ii) for any $x \in \overline{\mathcal{R}}$, the linear hull of $\{\chi_S \,:\, S \in \mathcal{T}^{\mu}_{x}\}$ is $\mathbb{R}^N$, where $\mathcal{T}^{\mu}_{x} \equiv \{S \subseteq N \,:\, \mu(S) = \sum_{i \in S} x_i\}$,

consider a fixed $x \in \overline{\mathcal{R}} = \mathsf{ext}\, C(\mu)$ and realize that the vectors in the set $\mathcal{V} := \{\chi_S \in \mathbb{R}^N \,:\, S \in \mathcal{T}^{\mu}_{x}\} \cup \{-\chi_N\}$ belong to the (inner) normal cone of (the least face of $C(\mu)$ containing) the vector x, which is defined by

$$N_x \;:=\; \{\, v \in \mathbb{R}^N \;:\; \quad \forall\, y \in C(\mu) \;\; \langle v, y\rangle \geq \langle v, x\rangle \,\} \;\equiv\; \{\, v \in \mathbb{R}^N \;:\; \min_{y \in C(\mu)} \langle v, y\rangle = \langle v, x\rangle \,\}.$$

Indeed, for $v = \chi_S$, $S \in \mathcal{T}^{\mu}_{x}$, one has

$$\langle v, y\rangle = \sum_{i \in S} y_i \geq \mu(S) = \sum_{i \in S} x_i = \langle v, x\rangle \quad \text{for any } y \in C(\mu).$$

The cone N_x is the conic hull of $\mathcal{V}$, which observation can be derived from Farkas's lemma: if $t \in \mathbb{R}^N$ is not in the conic hull of $\mathcal{V}$ then $w \in \mathbb{R}^N$ exists such that $\langle v, w\rangle \geq 0$ for any $v \in \mathcal{V}$ while $\langle t, w\rangle < 0$. The former condition allows one to show that $y^{\varepsilon} := x + \varepsilon \cdot w$ belong to $C(\mu)$ for $0 < \varepsilon < \varepsilon_0$ with some small ε_0. Indeed, if $S \in \mathcal{T}^{\mu}_{x}$ then

$$\sum_{i \in S} y^{\varepsilon}_i = \sum_{i \in S} x_i + \varepsilon \cdot \sum_{i \in S} w_i = \mu(S) + \varepsilon \cdot \underbrace{\langle \chi_S, w\rangle}_{\geq 0} \geq \mu(S)$$

and in case $S \notin \mathcal{T}^{\mu}_{x}$ one has

$$\lim_{\varepsilon \to 0} \sum_{i \in S} y^{\varepsilon}_i - \mu(S) = \sum_{i \in S} x_i - \mu(S) + \lim_{\varepsilon \to 0} \varepsilon \cdot \sum_{i \in S} w_i = \sum_{i \in S} x_i - \mu(S) > 0\,.$$

The latter condition says $\langle t, y^{\varepsilon}\rangle - \langle t, x\rangle = \langle t, y^{\varepsilon} - x\rangle = \varepsilon \cdot \langle t, w\rangle < 0$, implying that $t \notin N_x$ (as $\exists\, y^{\varepsilon} \in C(\mu) \,:\, \langle t, y^{\varepsilon}\rangle < \langle t, x\rangle$).

The next observation is that, for any $x \in C(\mu)$, x is a vertex of $C(\mu)$ iff its normal cone N_x is full-dimensional. This result holds for any polytope $P \subseteq \mathbb{R}^N$ in place of $C(\mu)$. To see why this is the case the reader is advised to consult [22, § 7.1] for basic facts about the collection of normal cones for a polytope P, named the *normal fan* of the polytope. One needs to realize that the lattice of normal cones is anti-isomorphic to the face-lattice of P. The latter means, for $x, y \in P$, that one has

$$N_y \subseteq N_x \;\Leftrightarrow\; F[y] \supseteq F[x] \qquad \text{where } F[x] \text{ denotes the least face of } P \text{ containing } x \in P.$$

To this end realize that, for any $v \in N_x$ and $z \in P$, $\langle v, x\rangle = \langle v, z\rangle$ iff $v \in N_z$, which allows one to observe

$$F[x] := \bigcap_{v \in N_x} \{z \in P \,:\, \langle v, x\rangle = \langle v, z\rangle\} = \bigcap_{v \in N_x} \{z \in P \,:\, v \in N_z\} = \{z \in P \,:\, N_x \subseteq N_z\}\,.$$

Hence we get

$$x \text{ is a vertex of } P \;\Leftrightarrow\; F[x] = \{x\} \;\Leftrightarrow\; N_x \text{ is a maximal cone} \;\Leftrightarrow\; N_x \text{ has the dimension } |N|\,.$$

Since the linear hull of the normal cone N_x for a vertex $x \in \overline{\mathcal{R}}$ is the linear hull of $\{\chi_S \,:\, S \in \mathcal{T}^{\mu}_{x}\}$ the condition (ii) for x is implied by the last observation.

It follows from above arguments that any min-representation $\mathcal{R} \subseteq \mathsf{ext}\, C(\mu)$ is regular. Conversely, given a regular min-representation $\mathcal{R}$ of μ, its elements belong to the core of μ and the regularity condition (ii) for $x \in \mathcal{R}$ implies that the respective normal cone N_x is full-dimensional, which happens only in case x is a vertex of $C(\mu)$. $\square$

Appendix A.2. Proof of Proposition 2

Recall that, given an exact game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$, we introduced in Definition 5 a *refinement relation* for core-based min-representations $\mathcal{R}, \mathcal{L} \subseteq C(\mu)$ of μ:

$$\mathcal{R} \text{ refines } \mathcal{L} \;\Leftrightarrow\; [\;\forall\, x \in \mathcal{R} \quad \exists\, y \in \mathcal{L} \qquad \mathcal{T}^{\mu}_{x} \subseteq \mathcal{T}^{\mu}_{y}\;] \quad \text{where } \mathcal{T}^{\mu}_{x} \;\equiv\; \{\, S \subseteq N \,:\, \mu(S) = \sum_{i \in S} x_i \,\} \text{ for any } x \in C(\mu).$$

The min-representations $\mathcal{R}$ and $\mathcal{L}$ are called *equally fine* if they refine each other. The tightness structure of a min-representation $\mathcal{R}$ is $\{\, \mathcal{T}^{\mu}_{x} \,:\, x \in \mathcal{R}\,\}$. We are going to prove the following.

Proposition 2: Given an exact game $\mu \in \mathbb{R}^{\mathcal{P}(N)}$, the set of tightness structures of its core-based min-representations is a lattice with respect to the refinement relation and the equivalence of being equally fine. Formally, elements of the lattice are the equivalence classes ordered by the refinement relation. The largest element in the lattice is the *coarsest tightness structure* of the standard min-representation. The *finest tightness structure* can be constructed from the standard min-representation $\overline{\mathcal{R}} = \mathrm{ext}\, C(\mu)$ as follows:

[i] for any $S \subseteq N$ consider the set $\overline{\mathcal{R}}[S] := \{x \in \overline{\mathcal{R}} \,:\, S \in \mathcal{T}^{\mu}_{x}\}$ of vertices of $C(\mu)$ having S in the tightness class and define $y^S \in C(\mu)$ as their arithmetic mean:

$$y^S \;:=\; \sum_{x \in \overline{\mathcal{R}}[S]} \frac{1}{|\overline{\mathcal{R}}[S]\,|} \cdot x\,; \qquad \text{note that } \mathcal{T}^{\mu}_{y} \,=\, \bigcap_{x \in \overline{\mathcal{R}}[S]} \mathcal{T}^{\mu}_{x} \quad \text{for } y = y^S\,.$$

[ii] prune (the tightness structure of) the min-representation $\{y^S \,:\, S \subseteq N\}$ of μ by the removal of ($y^S = y$ with) non-maximal classes $\mathcal{T}^{\mu}_{y}$ with respect to inclusion.

Proof. The fact that the refinement relation is reflexive and transitive is evident. In particular, the relation of being equally fine is an equivalence relation (on the collection of tightness structures of core-based min-representations of μ). Recall that every equivalence class can uniquely be described by the respective pruned tightness structure, that is, $\{\mathcal{T}^{\mu}_{x} \,:\, x \in \mathcal{R}\}$, where the tightness classes $\mathcal{T}^{\mu}_{x}$ are inclusion incomparable (see § 4). To show that this finite poset (of pruned tightness structures) is a lattice it is enough to observe that

- there is the largest element in the poset,
- every pair of tightness structures has the infimum.

The largest element in the poset appears to be the tightness structure of the standard min-representation. Indeed, for any core-based min-representation $\mathcal{R}$ and $x \in \mathcal{R} \subseteq C(\mu)$, one can write $x = \sum_{y \in \overline{\mathcal{R}}} k_y \cdot y$ with $k_y \geq 0$ and $\sum_{y \in \overline{\mathcal{R}}} k_y = 1$. Hence, for any $S \in \mathcal{T}^{\mu}_{x}$, one has

$$0 = \sum_{i \in S} x_i - \mu(S) = \sum_{y \in \overline{\mathcal{R}}:\, k_y > 0} k_y \cdot \underbrace{\left[\, \sum_{i \in S} y_i - \mu(S) \right]}_{\geq 0},$$

which implies that $\mathcal{T}^{\mu}_{x} \subseteq \mathcal{T}^{\mu}_{y}$ for any $y \in \overline{\mathcal{R}}$ with $k_y > 0$. Thus, the tightness structure of $\overline{\mathcal{R}}$ is coarser than the one of $\mathcal{R}$.

Given two core-based min-representations $\mathcal{R}$ and $\mathcal{L}$ of μ, it is evident that

$$\{\, \mathcal{T}^{\mu}_{z} \equiv \mathcal{T}^{\mu}_{x} \cap \mathcal{T}^{\mu}_{y} \,:\, z = \frac{1}{2} \cdot x + \frac{1}{2} \cdot y,\; x \in \mathcal{R},\; y \in \mathcal{L}\,\}$$

defines a common refinement of the tightness structures of $\mathcal{R}$ and $\mathcal{L}$, and, thus, yielding their infimum.

To verify the last claim about the constructed finest tightness structure it is enough to show that it refines any tightness structure. Given a core-based min-representation $\mathcal{R}$, for any $S \subseteq N$, $x \in \mathcal{R}$ exists with $S \in \mathcal{T}^{\mu}_{x}$. Write $x = \sum_{z \in \mathcal{Z}} k_z \cdot z$ with $k_z > 0$, $\sum_{z \in \mathcal{Z}} k_z = 1$, where $\mathcal{Z} \subseteq \overline{\mathcal{R}} \equiv \mathrm{ext}\, C(\mu)$. Note that this implies $\mathcal{T}^{\mu}_{x} = \bigcap_{z \in \mathcal{Z}} \mathcal{T}^{\mu}_{z}$: for any $T \subseteq N$ one has

$$\sum_{i \in T} x_i - \mu(T) = \sum_{z \in \mathcal{Z}} k_z \cdot \sum_{i \in T} z_i - \sum_{z \in \mathcal{Z}} k_z \cdot \mu(T) = \sum_{z \in \mathcal{Z}} k_z \cdot \underbrace{(\sum_{i \in T} z_i - \mu(T))}_{\geq 0}$$

and the expression vanishes iff $\sum_{i\in T} z_i = \mu(T)$ for any $z \in \mathcal{Z}$. The verification of the fact $\mathcal{T}^{\mu}_{y} = \bigcap_{z\in\overline{\mathcal{R}}[S]} \mathcal{T}^{\mu}_{z}$ for $y = y^S$ is analogous. Therefore, for that fixed $S \subseteq N$ and $x \in \mathcal{R}$ with $S \in \mathcal{T}^{\mu}_{x}$, the inclusion $\mathcal{Z} \subseteq \overline{\mathcal{R}}[S]$ gives

$$\mathcal{T}^{\mu}_{y} = \bigcap_{z\in\overline{\mathcal{R}}[S]} \mathcal{T}^{\mu}_{z} \subseteq \bigcap_{z\in\mathcal{Z}} \mathcal{T}^{\mu}_{z} = \mathcal{T}^{\mu}_{x}\,.$$

In other words, for every $y = y^S$, there exists $x \in \mathcal{R}$ such that $\mathcal{T}^{\mu}_{y} \subseteq \mathcal{T}^{\mu}_{x}$, which fact we wished to verify. □

Appendix A.3. Proof of Proposition 3

Recall that it was mentioned in § 5.2 that additional "unwanted" non-zero solutions to the equation system

(a) $\forall\,\gamma \in \Gamma\ \ \forall\, i \in N$ with $\{i\} \in \mathcal{T}_{\gamma}$ $\qquad \mathsf{y}(\gamma, i) = 0$,

(b) $\forall\, S \subseteq N$, $|S| \geq 2$, $\forall\,\gamma,\rho \in \Gamma$ with $S \in \mathcal{T}_{\gamma} \cap \mathcal{T}_{\rho}$ $\qquad \sum_{i\in S} \mathsf{y}(\gamma, i) = \sum_{i\in S} \mathsf{y}(\rho, i)$,

may exist, where, by (5) in § 5.1,

$$\mathcal{T}_{\gamma} \;\equiv\; \{\, S \subseteq N \,:\, \mu(S) = \sum_{i\in S} \mathsf{x}(\gamma, i)\,\} \quad \text{for any } \gamma \in \Gamma\,.$$

Specifically, if one denotes

$$L_{\gamma} \equiv \mathrm{Lin}\,\{\chi_S \,:\, S \in \mathcal{T}_{\gamma}\} \quad \text{for any } \gamma \in \Gamma\,,$$

and, for some $\gamma^* \in \Gamma$, a vector $v \in \mathbb{R}^N$ belongs to the orthogonal complement of L_{γ^*} in $\mathbb{R}^N$, then the formula

$$\mathsf{y}(\gamma, i) \;:=\; \begin{cases} v_i & \text{if } \gamma = \gamma^*, \\ 0 & \text{if } \gamma \neq \gamma^*, \end{cases} \qquad \text{for any } \gamma \in \Gamma,\ i \in N, \tag{6}$$

also gives a solution to (a)-(b). We are going to prove the following.

Proposition 3: Given $\mathsf{z} \in \mathbb{R}^{\Gamma\times N}$ satisfying (a)-(b) a uniquely determined ℓ-standardized game $v \in \mathbb{R}^{\mathcal{P}(N)}$ exists such that

$$\forall\,\gamma \in \Gamma\ \ \forall\, S \in \mathcal{T}_{\gamma} \qquad v(S) = \sum_{i\in S} \mathsf{z}(\gamma, i)\,. \tag{7}$$

The mapping $\mathcal{A} : \mathsf{z} \in R_{\Gamma} \subseteq \mathbb{R}^{\Gamma\times N} \mapsto v \in \mathbb{R}^{\mathcal{P}(N)}$, where R_{Γ} is the space of solutions to (a)-(b), is linear and the set $\mathcal{A}_{-1}(\mathbf{0})$ coincides with the linear hull of solutions to (a)-(b) of type (6). The dimension of $\mathcal{A}_{-1}(\mathbf{0})$ is

$$\sum_{\gamma\in\Gamma} \mathrm{codim}\,(L_{\gamma}) \qquad \text{where, for any } \gamma \in \Gamma, \quad \mathrm{codim}\,(L_{\gamma}) := |N| - \dim(L_{\gamma}) = \dim(L^{\perp}_{\gamma})$$

is the dimension of the orthogonal complement $L^{\perp}_{\gamma}$ of the space L_{γ} in $\mathbb{R}^N$.

Proof. Realize that (a)-(b) for z together imply the *consistency condition*

$$\forall\, S \subseteq N \ \ \forall\,\gamma,\rho \in \Gamma \ \text{ with } S \in \mathcal{T}_{\gamma} \cap \mathcal{T}_{\rho} \qquad \sum_{i\in S} \mathsf{z}(\gamma, i) = \sum_{i\in S} \mathsf{z}(\rho, i)\,.$$

Since the min-representation condition (see § 5.1)

$$\forall\, S \subseteq N \qquad \mu(S) = \min_{\gamma\in\Gamma} \sum_{i\in S} \mathsf{x}(\gamma, i)\,. \tag{4}$$

gives $\mathcal{P}(N) = \bigcup_{\gamma\in\Gamma} \mathcal{T}_{\gamma}$, one can correctly define $v \equiv \mathcal{A}(\mathsf{z})$ using (7). This game v is uniquely determined through (7). The function $\mathcal{A} :\ \mathsf{z} \in R_{\Gamma} \mapsto v \in \mathbb{R}^{\mathcal{P}(N)}$ is linear by definition. Moreover, the condition (a) for z implies that $v = \mathcal{A}(\mathsf{z})$ must be ℓ-standardized.

It follows from (7) that z is mapped by $\mathcal{A}$ to a zero game iff, for any $\gamma \in \Gamma$, the row-vector $\mathsf{z}(\gamma, *)$ belongs to the orthogonal complement of L_{γ} in $\mathbb{R}^N$. Thus, z can be written as the sum of solutions of type (6); these are linearly independent for different rows. This implies the claim about the dimension of $\mathcal{A}_{-1}(\mathbf{0})$. □

Appendix A.4. Proof of Proposition 4

Note that the definitions of concepts used in the proof below are recalled in Appendix A.3. The only additional concept is that of the *essential dimension*:

$$\text{esdim}\,(\mathcal{R}) \equiv \dim(R_\Gamma) - \dim(\mathcal{A}_{-1}(\mathbf{0})) = \dim(R_\Gamma) - \sum_{\gamma\in\Gamma} \text{codim}\,(L_\gamma)$$

for a min-representation $\mathcal{R}$; see § 5.2. Let us recall what we are going to prove.

Proposition 4: A non-zero ℓ-standardized exact game $\mu \in \mathsf{E}_\ell(N)$ is *extreme* in $\mathsf{E}_\ell(N)$ iff, for each core-based min-representation of μ, arranged in an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ with (4), the essential dimension is 1.

Note that an equivalent formulation of the condition from Proposition 4 is that the dimension of the linear space R_Γ of solutions to the linear equation system (a)-(b) is $1 + \sum_{\gamma\in\Gamma} \text{codim}\,(L_\gamma)$.

Proof. To verify the sufficiency of the condition from Proposition 4 we show it implies that whenever $\mu = \mu^1 + \mu^2$ where $\mu^1 \in \mathsf{E}_\ell(N)$ and $\mu^2 \in \mathsf{E}_\ell(N)$ is non-zero, then $k \geq 0$ exists with $\mu^1 = k \cdot \mu^2$. To this end we put

$$\mathcal{R} \;:=\; \mathsf{ext}\, C(\mu^1) \oplus \mathsf{ext}\, C(\mu^2)\,,$$

where $\oplus$ denotes the Minkowski sum (see § 2). Since, by Proposition 1, $\mathsf{ext}\, C(\mu^j)$ is a core-based min-representation of μ^j, $j = 1, 2$, one can easily observe that $\mathcal{R}$ is a core-based min-representation of μ.

Assume that $\mathcal{R}$ is arranged in the form of an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ satisfying (4). Consider the respective equation system (a)-(b). We already know, by Proposition 3, that $\sum_{\gamma\in\Gamma} \text{codim}\,(L_\gamma)$ linearly independent solutions to (a)-(b) exist in $\mathcal{A}_{-1}(\mathbf{0})$.

The idea is to construct two other solutions to (a)-(b), which do not belong to $\mathcal{A}_{-1}(\mathbf{0})$. Recall we assume that $\mu = \mu^1 + \mu^2$ with $\mu^1, \mu^2 \in \mathsf{E}_\ell(N)$, $\mu^2 \neq \mathbf{0}$. Realize that every $x \in \mathcal{R}$ is the sum of two vertices of corresponding cores: $x = x^1 + x^2$ where $x^1 \in \mathsf{ext}\, C(\mu^1)$ and $x^2 \in \mathsf{ext}\, C(\mu^2)$. To observe that one has $S \in \mathcal{T}^\mu_x \;\Rightarrow\; S \in \mathcal{T}^{\mu^1}_{x^1} \cap \mathcal{T}^{\mu^2}_{x^2}$ in this situation write

$$0 = \sum_{i\in S} x_i - \mu(S) = \sum_{i\in S} x^1_i + \sum_{i\in S} x^2_i - \mu^1(S) - \mu^2(S) = \underbrace{\left(\sum_{i\in S} x^1_i - \mu^1(S)\right)}_{\geq 0} + \underbrace{\left(\sum_{i\in S} x^2_i - \mu^2(S)\right)}_{\geq 0}$$

and realize that both non-negative summands must vanish. Denote by Υ^1 the index set for the standard array min-representation of μ^1, analogously Υ^2 for μ^2. Thus, for any $\gamma \in \Gamma$, rows $\gamma^1 \in \Upsilon^1$ and $\gamma^2 \in \Upsilon^2$ exist such that

$$\mathsf{x}(\gamma, i) = \mathsf{x}^1(\gamma^1, i) + \mathsf{x}^2(\gamma^2, i) \qquad \text{for any } i \in N.$$

We fix the pair (γ^1, γ^2) for any $\gamma \in \Gamma$ then. The desired different solutions to (a)-(b) with Γ will be

$$\mathsf{y}^1(\gamma, i) := \mathsf{x}^1(\gamma^1, i), \quad \mathsf{y}^2(\gamma, i) := \mathsf{x}^2(\gamma^2, i) \qquad \text{for any } \gamma \in \Gamma \text{ and } i \in N.$$

To observe that y^1 solves (a)-(b) use the inclusion $\mathcal{T}_\gamma \subseteq \mathcal{T}_{\gamma^1}$ (where the latter is the tightness class relative to μ^1) and the fact that x^1 solves (a)-(b) in space $\mathbb{R}^{\Upsilon^1\times N}$; analogously with y^2. Similarly, it is not difficult to observe that the linear map $\mathcal{A}$ from Proposition 3 transforms y^1 to μ^1 and y^2 to μ^2. Because $\mathsf{y}^2 \notin \mathcal{A}_{-1}(\mathbf{0})$ the linear hull of $\mathcal{A}_{-1}(\mathbf{0}) \cup \{\mathsf{y}^2\}$ has the dimension at least $1 + \sum_{\gamma\in\Gamma} \text{codim}\,(L_\gamma)$. Hence, the assumption implies that $\text{Lin}\left(\mathcal{A}_{-1}(\mathbf{0}) \cup \{\mathsf{y}^2\}\right)$ is the whole space R_Γ of solutions, which forces $\mathsf{y}^1 = k \cdot \mathsf{y}^2 + \sum_{\mathsf{z}\in A'} \ell_{\mathsf{z}} \cdot \mathsf{z}$, for some finite $A' \subseteq \mathcal{A}_{-1}(\mathbf{0})$ and real coefficients $k, \ell_{\mathsf{z}} \in \mathbb{R}$. The linearity of the mapping $\mathcal{A}$ from Proposition 3 and the fact that it maps elements of A' to $\mathbf{0}$ then gives $\mu^1 = k \cdot \mu^2$. Since both $\mu^1(N) \geq 0$ and $\mu^2(N) > 0$, one has $k \geq 0$, which concludes the proof of the sufficiency.

To verify the necessity of the condition from Proposition 4 it is enough to show that its negation implies that μ is a convex combination of $\mu^1, \mu^2 \in \mathsf{E}_\ell(N)$ none of which is a multiple of μ. For this purpose assume that a core-based min-representation $\emptyset \neq \mathcal{R} \subseteq C(\mu)$ exists, arranged in the form of an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ satisfying (4), such that the linear space of solutions to (a)-(b) has the dimension different from $1 + \sum_{\gamma\in\Gamma} \text{codim}\,(L_\gamma)$.

Consider the linear mapping $\mathcal{A}$ from Proposition 3; by (4), $\mathcal{A}(\mathsf{x}) = \mu \neq \mathbf{0}$, which implies that $\mathsf{x} \notin \mathcal{A}_{-1}(\mathbf{0})$. Let B denote the linear hull of $\mathcal{A}_{-1}(\mathbf{0}) \cup \{\mathsf{x}\}$; its dimension is $1 + \sum_{\gamma\in\Gamma} \operatorname{codim}(L_\gamma)$. The vectors in B are solutions to (a)-(b) and the assumption implies that a solution $\mathsf{y} \in R_\Gamma \subseteq \mathbb{R}^{\Gamma\times N}$ to (a)-(b) exists which is outside B: $\mathsf{y} \notin B$.

Consider the line L in $\mathbb{R}^{\Gamma\times N}$ passing through y and x, namely the vectors

$$\mathsf{z}^\varepsilon := (1-\varepsilon)\cdot \mathsf{x} + \varepsilon\cdot \mathsf{y} \qquad \text{for any } \varepsilon \in \mathbb{R}\,.$$

Observe that L does not contain a vector in $\mathcal{A}_{-1}(\mathbf{0})$ as otherwise, two different vectors of L belongs to B implying that the whole line L, including y, belongs to B. Moreover, the only vector of L which belongs to B is x, as otherwise, by an analogous consideration, the whole line, including y, belongs to B. As vectors in L satisfy (a)-(b), by Proposition 3, ℓ-standardized games v^ε, $\varepsilon \in \mathbb{R}$, exist such that

$$\forall\, \varepsilon \in \mathbb{R} \quad \forall\, \gamma \in \Gamma \text{ with } S \in \mathcal{T}_\gamma \qquad \sum_{i\in S} \mathsf{z}^\varepsilon(\gamma, i) = v^\varepsilon(S)\,.$$

The next step is to show that, for sufficiently small $|\varepsilon|$, one has

$$\forall\, \gamma \in \Gamma \text{ and } S \subseteq N \text{ with } S \notin \mathcal{T}_\gamma \qquad \sum_{i\in S} \mathsf{z}^\varepsilon(\gamma, i) > v^\varepsilon(S)\,, \tag{A.1}$$

which implies, for those small $|\varepsilon|$, that

$$v^\varepsilon(S) = \min_{\gamma\in\Gamma} \sum_{i\in S} \mathsf{z}^\varepsilon(\gamma, i) \qquad \text{for any } S \subseteq N.$$

In particular, v^ε has a core-based min-representation and, by Proposition 1, one has $v^\varepsilon \in \mathsf{E}_\ell(N)$ for those small $|\varepsilon|$.

To ensure (A.1) for small $|\varepsilon|$, consider a fixed $\gamma \in \Gamma$ and $S \subseteq N$, $S \notin \mathcal{T}_\gamma$, and choose $\pi \in \Gamma$ such that $S \in \mathcal{T}_\pi$, by (4). The definitions of $\mathcal{T}_\gamma$ and $\mathcal{T}_\pi$ then imply

$$0 \;<\; \sum_{i\in S} \mathsf{x}(\gamma, i) \,-\, \mu(S) = \sum_{i\in S} \mathsf{x}(\gamma, i) \,-\, \sum_{i\in S} \mathsf{x}(\pi, i)\,.$$

This allows us to write

$$\begin{aligned}\sum_{i\in S} \mathsf{z}^\varepsilon(\gamma, i) \,-\, v^\varepsilon(S) \;&=\; \sum_{i\in S} \mathsf{z}^\varepsilon(\gamma, i) \,-\, \sum_{i\in S} \mathsf{z}^\varepsilon(\pi, i)\\ &=\; (1-\varepsilon)\cdot \underbrace{\left(\sum_{i\in S} \mathsf{x}(\gamma, i) - \sum_{i\in S} \mathsf{x}(\pi, i)\right)}_{>0} \;+\; \varepsilon\cdot\left(\sum_{i\in S} \mathsf{y}(\gamma, i) - \sum_{i\in S} \mathsf{y}(\pi, i)\right),\end{aligned}$$

and observe that the limit of this expression with ε tending to zero is positive. Therefore, after considering all pairs (γ, S), $\gamma \in \Gamma$, $S \notin \mathcal{T}_\gamma$, (A.1) is ensured for sufficiently small $|\varepsilon|$.

Thus, there exists $0 < \varepsilon$ such that both $\kappa := (1-\varepsilon)\cdot\mu + \varepsilon\cdot v^1$ and $\lambda := (1+\varepsilon)\cdot\mu - \varepsilon\cdot v^1$ belong to $\mathsf{E}_\ell(N)$. Clearly, $\mu = \frac{1}{2}\cdot\kappa + \frac{1}{2}\cdot\lambda$. Neither κ nor λ is a multiple of μ. To show that assume for a contradiction that $\kappa = k\cdot\mu$ for $k \geq 0$. Consider the arrays $k\cdot\mathsf{x}$ and z^ε in $\mathbb{R}^{\Gamma\times N}$ defining through (7) $k\cdot\mu$ and $v^\varepsilon = \kappa$, respectively. Hence, their difference $k\cdot\mathsf{x} - \mathsf{z}^\varepsilon$ belongs to the linear space $\mathcal{A}_{-1}(\mathbf{0})$. This means that $\mathsf{z}^\varepsilon \in B$ which contradicts the fact that the line L intersects B solely in the vector x. Since neither κ nor λ is a multiple of μ, the proof of the necessity of the condition from Proposition 4 is complete. □

Appendix A.5. Proof of Proposition 5

Recall from § 5.1 that we can arrange any core-based min-representation of an exact game μ in the form of an array $\mathsf{x} \in \mathbb{R}^{\Gamma\times N}$ satisfying

$$\forall\, S \subseteq N \qquad \mu(S) = \min_{\gamma\in\Gamma} \sum_{i\in S} \mathsf{x}(\gamma, i) \tag{4}$$

and that each row is then assigned a special linear subspace of $\mathbb{R}^N$ by

$$L_\gamma \;\equiv\; \mathrm{Lin}\{\chi_S \;:\; S\in\mathcal{T}_\gamma\} \quad \text{where } \mathcal{T}_\gamma \;\equiv\; \{S\subseteq N \;:\; \mu(S)=\textstyle\sum_{i\in S}\mathsf{x}(\gamma,i)\} \;\text{ for } \gamma\in\Gamma. \tag{5}$$

The essential dimension of a core-based min-representation $\mathcal{R}$ of μ is then

$$\mathrm{esdim}\,(\mathcal{R}) \;\equiv\; \dim(R_\Gamma)-\dim(\mathcal{A}_{-1}(\mathbf{0}))\,,$$

where R_Γ is the space of solutions $\mathsf{y}\in\mathbb{R}^{\Gamma\times N}$ to (a)-(b) and $\mathcal{A}_{-1}(\mathbf{0})$ is the space of dummy solutions to (a)-(b) from Proposition 3. Let us recall what we are going to prove.

Proposition 5: Let $\mathcal{R}$ and $\mathcal{L}$ be core-based min-representations of a non-zero exact ℓ-standardized game μ and the tightness structure of $\mathcal{R}$ refines the tightness structure of $\mathcal{L}$, which means that

$$\forall\, x\in\mathcal{R}\quad \exists\, y\in\mathcal{L} \qquad \mathcal{T}^\mu_x\subseteq\mathcal{T}^\mu_y\,.$$

Then one has

$$1\leq \mathrm{esdim}\,(\mathcal{L})\leq \mathrm{esdim}\,(\mathcal{R})\leq\; 2^{|N|}-|N|-1\,.$$

Hence, the essential dimension of equally fine min-representations is the same.

Proof. The inequality $1\leq\mathrm{esdim}\,(\mathcal{L})$ follows from the definition of the essential dimension and the assumption that μ is non-zero (see the analysis in § 5.2). We are going to show that $\mathrm{esdim}\,(\mathcal{L})\leq\mathrm{esdim}\,(\mathcal{R})$ whenever either of the following two cases occurs:

I. $\mathcal{L}=\mathcal{R}\cup\{v\}$ where $v\in C(\mu)$,

II. $\mathcal{L}=\mathcal{R}\setminus\{w\}$ while $v\in\mathcal{L}$ exists with $\mathcal{T}^\mu_w\subseteq\mathcal{T}^\mu_v$.

This is enough because whenever the tightness structure of $\mathcal{R}$ refines that of $\mathcal{L}$ then $\mathcal{L}$ can be obtained from $\mathcal{R}$ by a series of operations of type I. and II.

A preparatory consideration is as follows. Assume that $\mathcal{R}$ is arranged in a real array $\mathsf{x}\in\mathbb{R}^{\Gamma\times N}$, as described in § 5.1. Then the linear space $R_\Gamma\subseteq\mathbb{R}^{\Gamma\times N}$ of solutions to (a)-(b) has the following orthogonal decomposition:

$$R_\Gamma=\mathcal{A}_{-1}(\mathbf{0})\oplus T_\Gamma \quad \text{where } \mathcal{A}_{-1}(\mathbf{0}) \text{ is the space from Proposition 3}$$
$$\text{and}\quad T_\Gamma:=\{\,\mathsf{y}\in\mathbb{R}^{\Gamma\times N}\;:\;\mathsf{y}\text{ satisfies (a)-(b) } \&\ \mathsf{y}(\gamma,*)\in L_\gamma \text{ for any } \gamma\in\Gamma\,\}$$

is the space of *pure solutions*. Indeed, given a solution $\mathsf{y}\in\mathbb{R}^{\Gamma\times N}$, for any $\gamma\in\Gamma$, the respective row-vector in $\mathbb{R}^N$ has unique orthogonal decomposition $\mathsf{y}(\gamma,*)=\mathsf{y}^0(\gamma,*)+\mathsf{y}'(\gamma,*)$ with $\mathsf{y}^0(\gamma,*)\in L^\perp_\gamma$ and $\mathsf{y}'(\gamma,*)\in L_\gamma$. The fact

$$0=\langle\chi_S,\mathsf{y}^0(\gamma,*)\rangle=\sum_{i\in S}\mathsf{y}^0(\gamma,i)\quad\Rightarrow\quad\sum_{i\in S}\mathsf{y}'(\gamma,i)=\sum_{i\in S}\mathsf{y}(\gamma,i)\quad\text{for } S\in\mathcal{T}_\gamma$$

allows one to verify that the both vectors $\mathsf{y}^0,\mathsf{y}'\in\mathbb{R}^{\Gamma\times N}$ satisfy (a)-(b). They are orthogonal in $\mathbb{R}^{\Gamma\times N}$ because the scalar product in $\mathbb{R}^{\Gamma\times N}$ is the sum of scalar products in $\mathbb{R}^N$ for the rows. The consequence of this fact is that the essential dimension is nothing but the dimension of the space of pure solutions: $\mathrm{esdim}\,(\mathcal{R})=\dim(T_\Gamma)$.

In case I. arrange $\mathcal{L}$ into a real array $\mathsf{x}\in\mathbb{R}^{\Sigma\times N}$ (as described in § 5.1) and assume that $\Sigma=\Gamma\cup\{\sigma\}$ where $v=\mathsf{x}(\sigma,*)\in C(\mu)$; that is, the sub-array x_Γ is an arrangement of $\mathcal{R}$. To show $\dim(T_\Sigma)\leq\dim(T_\Gamma)$ we consider a linear mapping from T_Σ to T_Γ which ascribes to $\mathsf{y}\in T_\Sigma\subseteq\mathbb{R}^{\Sigma\times N}$ its restriction $\mathsf{y}_\Gamma\in\mathbb{R}^{\Gamma\times N}$ to rows in Γ. It is evident that $\mathsf{y}_\Gamma\in T_\Gamma$ and it remains to show that the mapping is injective, which means, if $\mathsf{y}^1,\mathsf{y}^2\in T_\Sigma$ and $\mathsf{y}^1_\Gamma=\mathsf{y}^2_\Gamma$ then $\mathsf{y}^1=\mathsf{y}^2$. This reduces to showing $\mathsf{y}^1(\sigma,*)=\mathsf{y}^2(\sigma,*)$. By the definition of T_Σ we know that both these vectors belong to L_σ, which gives $\mathsf{y}^1(\sigma,*)-\mathsf{y}^2(\sigma,*)\in L_\sigma$. To conclude that the vectors coincide it is enough to show $\mathsf{y}^1(\sigma,*)-\mathsf{y}^2(\sigma,*)\in L^\perp_\sigma$. That means, for any $S\in\mathcal{T}_\sigma$, to verify $\langle\chi_S\,,\,\mathsf{y}^1(\sigma,*)-\mathsf{y}^2(\sigma,*)\rangle=0$, which can be re-written in the form $\sum_{i\in S}\mathsf{y}^1(\sigma,i)=\sum_{i\in S}\mathsf{y}^2(\sigma,i)$. The condition (4) for Γ implies there exists $\gamma\in\Gamma$ such that $S\in\mathcal{T}_\gamma$; thus, $S\in\mathcal{T}_\sigma\cap\mathcal{T}_\gamma$ and by the consistency conditions (b) for $\mathsf{y}^1,\mathsf{y}^2\in T_\Sigma$ and the assumption $\mathsf{y}^1_\Gamma=\mathsf{y}^2_\Gamma$ write

$$\sum_{i\in S}\mathsf{y}^1(\sigma,i)=\sum_{i\in S}\mathsf{y}^1(\gamma,i)=\sum_{i\in S}\mathsf{y}^2(\gamma,i)=\sum_{i\in S}\mathsf{y}^2(\sigma,i)\,,$$

which gives what was desired. We have shown $\dim(T_\Sigma) \le \dim(T_\Gamma)$ in case I.

To cover the case II. it is enough to imagine $\mathcal{L}$ indexed by Γ and $\mathcal{R}$ by $\Sigma = \Gamma \cup \{\sigma\}$ with $w = \mathsf{x}(\sigma, *) \in \mathcal{R}$ and to extend the above observation as follows: whenever $\pi \in \Gamma$ exists such that $\mathcal{T}_\sigma \subseteq \mathcal{T}_\pi$ then $\dim(T_\Sigma) = \dim(T_\Gamma)$.

To this end we show that the restriction mapping from T_Σ to T_Γ is surjective in this case, which means that any $\mathsf{z} \in T_\Gamma$ is the restriction of some $\mathsf{y} \in T_\Sigma$. Consider the vector $\mathsf{z}(\pi, *) \in \mathbb{R}^N$ and its orthogonal decomposition (with respect to L_σ and its orthogonal complement $L_\sigma^\perp$ in $\mathbb{R}^N$):

$$\mathsf{z}(\pi, *) = z^0 + z' \qquad \text{where } z^0 \in L_\sigma^\perp \text{ and } z' \in L_\sigma\,.$$

Put $\mathsf{y}(\sigma, i) = z'_i$ for any $i \in N$ while $\mathsf{y}(\gamma, i) = z(\gamma, i)$ for $\gamma \in \Gamma$ and $i \in N$. Hence, $\mathsf{y}(\gamma, *) \in L_\gamma$, $\gamma \in \Gamma$, and $\mathsf{y}(\sigma, *) \in L_\sigma$ by the construction. Because z satisfies (a)-(b) to show that y satisfies (a)-(b) one can restrict to the case that $S \in \mathcal{T}_\sigma \cap \mathcal{T}_\gamma$ for $\gamma \in \Gamma$ and verify $\sum_{i \in S} \mathsf{y}(\sigma, i) = \sum_{i \in S} \mathsf{y}(\gamma, i)$. The assumption $\mathcal{T}_\sigma \subseteq \mathcal{T}_\pi$ implies that $S \in \mathcal{T}_\pi \cap \mathcal{T}_\gamma$ and write using the fact that $\sum_{i \in S} z_i^0 = \langle \chi_S, z^0 \rangle = 0$ (since $S \in \mathcal{T}_\sigma$ and $z^0 \in L_\sigma^\perp$):

$$\sum_{i \in S} \mathsf{y}(\sigma, i) = \sum_{i \in S} z'_i = \sum_{i \in S} z_i^0 + \sum_{i \in S} z'_i = \sum_{i \in S} \mathsf{z}(\pi, i) = \sum_{i \in S} \mathsf{z}(\gamma, i) = \sum_{i \in S} \mathsf{y}(\gamma, i)\,.$$

Thus, one has $\dim(T_\Sigma) = \dim(T_\Gamma)$, which concludes the proof in the case II.

The upper bound for $\operatorname{esdim}(\mathcal{R}) = \dim(T_\Gamma)$ follows from the fact that the linear mapping $\mathcal{A}$ from Proposition 3 restricted to T_Γ is injective. Therefore, the dimension of T_Γ is at most the dimension of $\mathsf{E}_\ell(N)$. □

References

[1] A. Antonucci and F. Cuzzolin. Credal set approximation by lower probabilities: application to credal networks. In E. Hüllermeier, R. Kruse, and F. Hoffman, editors, *Lecture Notes in AI 6178: IPMU 2010*, pages 716–725. Springer, 2010.

[2] J. De Bock and G. de Cooman. Extreme lower previsions. *Journal of Mathematical Analysis and Applications*, 421:1042–1080, 2015.

[3] A. G. Bronevich and I. N. Rozenberg. The description of extreme 2-monotone measures. In F. Cozman, T. Denouex, S. Destercke, and T. Seidenfeld, editors, *Proceedings of ISIPTA'13*, pages 23–32. Université de Technologie de Compiègne, 2013.

[4] W. Chang, J. Cheng, J. J. Allaire, Y. Xie, and J. McPherson. *shiny: Web Application Framework for R*, 2017. R package version 1.0.5.

[5] P. Csóka, P. J.-J. Herings, and L. A. Kóczy. Balancedness conditions for exact games. *Mathematical Methods of Optimization Research*, 74:44–52, 2011.

[6] J. Derks and J. Kuipers. On the number of extreme points of the core of a transferable utility game. In P. Borm and H. Peters, editors, *Chapters in Game Theory in Honour of Stef Tijs*, pages 83–97. Kluwer, 2002.

[7] K. Fukuda and A. Prodon. Double description method revisited. *Combinatorics and Computer Science*, pages 91–111, 1996.

[8] C. J. Geyer, G. D. Meeden, and incorporates code from cddlib written by K. Fukuda. *rcdd: Computational Geometry*, 2017. R package version 1.2.

[9] J. Kuipers, D. Vermeulen, and M. Voorneveld. A generalization of the Shapley-Ichiishi result. *International Journal of Game Theory*, 39:585–602, 2010.

[10] E. Lohmann, P. Borm, and P. J.-J. Herings. Minimal exact balancedness. *Mathematical Social Science*, 64:127–135, 2012.

[11] S. Maass. Continuous linear representation of coherent lower previsions. In J. M. Bernard, T. Seidenfeld, and M. Zaffalon, editors, *Proceedings in Informatics 18: ISIPTA'03*, pages 372–382. Carleton Scientific, 2003.

[12] E. Miranda and I. Montes. Games solutions, probability transformations and the core. In A. Antonucci, G. Corani, I. Couso, and S. Destercke, editors, *JMLR Workshops and Conference Proceedings 62: ISIPTA 2017*, pages 217–228, 2017.

[13] A. Postnikov, V. Reiner, and L. Williams. Faces of generalized permutohedra. *Documenta Mathematica*, 13:207–273, 2008.

[14] E. Quaeghebeur and G. de Cooman. Extreme lower probabilities. *Fuzzy Sets and Systems*, 159:2163–2175, 2008.

[15] R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2016.

[16] J. Rosenmüller. *Game Theory: Stochastics, Information, Strategies and Cooperation*. Kluwer, Boston, 2000.

[17] M. Studený. *Probabilistic Conditional Independence Structures*. Springer, London, 2005.

[18] M. Studený and V. Kratochvíl. Linear core-based criterion for testing extreme exact games. In A. Antonucci, G. Corani, I. Couso, and S. Destercke, editors, *JMLR Workshops and Conference Proceedings 62: ISIPTA 2017*, pages 313–324, 2017.

[19] M. Studený and T. Kroupa. Core-based criterion for extreme supermodular functions. *Discrete Applied Mathematics*, 20:122–151, 2016.

[20] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.

[21] A. Wallner. Maximal number of vertices of polytopes defined by f-probabilities. In F. G. Cozman, R. Nau, and T. Seidenfeld, editors, *Proceedings of ISIPTA'05*, pages 126–139, 2005.

[22] G. M. Ziegler. *Lectures on Polytopes*. Springer, New York, 1995.